A framework for Kleene algebra with an embedded structure

(Preliminary Version)

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Abstract. This paper proposes a framework for Kleene algebra with embedded structure, which enable us to handle different kind of Kleene algebras such as Kleene algebra with tests and Kleene algebra with relations uniformly. This framework guarantees the existence of free algebra if the embedded structure satisfies certain conditions.

1 Introduction

Kozen [10] defined a Kleene algebra with tests to be a Kleene algebra with an embedded Boolean algebra. Desharnais [3] defined a Kleene algebra with relations to be a Kleene algebra with an embedded relation algebra. Necessity to consider Kleene algebras with another embedded structure may occur.

The common feature of Kleene algebras with tests and Kleene algebras with relations is the fact that their underlying idempotent semiring structure is shared. Paying attention to the feature of Kleene algebras with tests, [6, 7] introduced *slightly* more general definition of Kleene algebras with tests than the definition of Kozen. Due to the definition, [5] gave systematic free construction of Kleene algebras with tests from a pair of sets. The category of Kleene algebras with tests is the comma category (U_{BI}, U_{KI}) of the forgetful functors U_{BI} from the category **Bool** of Boolean algebras to the category **ISR** of idempotent semirings and U_{KI} from the category **Kleene** of Kleene algebra to **ISR**.

The free construction in [5] is captured using adjunction between the category **Set** of sets and the category **Bool**, adjunction between **Set** and the category **Kleene**, the forgetful functor from **Bool** to **ISR**, adjunction between **ISR** and **Kleene**, and coproducts in **Kleene**. We can observe that the key of the construction is an adjunction $F_{IK} \dashv U_{KI}$ between **ISR** and **Kleene**, and coproducts in **Kleene**. It does not seem for free construction to be very important that the embedded structure is Boolean algebra.

Parameterising **Bool** in the setting of [5], we provide a framework for Kleene algebras with embedded structure $\mathcal{F}_{-}$. For example, replacing **Bool** with **RA**, we obtain the category $\mathcal{F}_{\mathbf{RA}}$ of slightly generalised Kleene algebras with relations.

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We show that the similar free construction is also available under a few conditions. The conditions provide sufficient conditions for existence of free Kleene algebras with an embedded structure.

2 Comma categories

Using a notion of comma categories, a framework for a Kleene algebra with an embedded structure may be provided. We recall some basic of comma categories. For more details of category theory we refer to [1, 12].

Definition 1 (Comma category). Given categories and functors $E \xrightarrow{T} C \xleftarrow{S} D$, the *comma category* (T, S) has as objects all triples $\langle e, d, f\rangle$, with e is an object in E, d is an object in E, and $f\colon T(e) \to S(d)$, and as arrows from $\langle e, d, f\rangle$ to $\langle e', d', f'\rangle$ all pairs $\langle k, h\rangle$ of arrows $k\colon e \to e'$, $h\colon d \to d'$ such that $f' \circ T(k) = S(h) \circ f$.

Given a category D with coproducts, for two objects d_1 and d_2 in D, $d_1 + d_2$ denotes a coproduct in D of d_1 and d_2. For two arrows $f\colon d_1 \to d_2$ and $g\colon d_1' \to d_2'$, $f + g$ denotes a unique arrow in D from $d_1 + d_2$ to $d_1' + d_2'$ such that $(f + g) \circ i = i' \circ f$ and $(f + g) \circ j = j' \circ g$, where i and j are injections of coproduct $d_1 + d_2$, and i' and j' so are of $d_1' + d_2'$.

Theorem 1. *Given categories and functors $E \xrightarrow{T} C \xleftarrow{S} D$, assume that S has a left adjoint V and the category D has coproducts. Then the functor $\Upsilon\colon (T, S) \to E \times D$ which is defined by the mapping*

$$\langle e, d, f\rangle \mapsto (e, d), \quad \langle k, h\rangle \mapsto (k, h)$$

has left adjoint.

Proof. For each object c in C and d in D, the bijection from $D(V(c), d)$ to $C(c, S(d))$, constituting the adjunction $V \dashv S$, is denoted by $\rho_{c,d}$. The subscript c, d will be omitted. The functor from $E \times D$ to (T, S) which is defined by the mapping

$$(e, d) \mapsto \langle e, V(T(e)) + d, \rho(i)\rangle, \quad (k, h) \mapsto \langle k, V(T(k)) + h\rangle \ ,$$

where i is the first injection of the coproduct $V(T(e)) + d$, is a left adjoint to Υ.

3 Kleene algebras

In this section we recall some basic of Kleene algebras [8, 9] and related structures. [4] contains several examples of Kleene algebras.

Definition 2 (Kleene algebra). A *Kleene algebra* is a set K equipped with nullary operators 0, 1 and binary operators $+$, $\cdot$, and a unary operator *, where the tuple $(K, +, \cdot, 0, 1)$ is an idempotent semiring and these data satisfy the following:

$$\begin{aligned} 1 + (p \cdot p^*) &= p^* \\ 1 + (p^* \cdot p) &= p^* \\ p \cdot r \leq r &\implies p^* \cdot r \leq r \\ r \cdot p \leq r &\implies r \cdot p^* \leq r \end{aligned}$$

where $\leq$ refers to the natural partial order $p \leq q \overset{\text{def}}{\iff} p+q = q$. A Kleene algebra will be called *trivial* if $0 = 1$, otherwise, called *non-trivial.* The category of Kleene algebras and homomorphisms between them will be denoted by **Kleene**.

Remark 1. **Kleene** has binary coproducts.

The injections of a coproduct in **Kleene** are not always one-to-one. Trivial Kleene algebras have only one element. For each Kleene algebra K, there exists a unique Kleene algebra homomorphism from K to the trivial one. From a trivial Kleene algebra, there exists a Kleene algebra homomorphism if the target is also trivial one. So, the coproduct of a trivial Kleene algebra and a non-trivial one is a trivial one again. Then, we have an injection which is not one-to-one. This example is due to Wolfram Kahl.

A Kleene algebra $\mathbf{K}$ is called *integral* if it has no zero divisors, that is,

$$a \neq 0 \ \wedge \ b \neq 0 \implies a \cdot b \neq 0$$

holds for all $a, b \in K$. This notion is introduced in [4].

Proposition 1. *Let* $\mathbf{J} = (J, +_J, \cdot_J, ^{*_J}, 0_J, 1_J)$ *and* $\mathbf{K} = (K, +_K, \cdot_K, ^{*_K}, 0_K, 1_K)$ *be non-trivial Kleene algebras. If* $\mathbf{K}$ *is integral, then the following holds.*

(i) The mapping $f\colon K \to J$ *defined to be* $f(a) = 0_J$ *if* $a = 0_K$*, and otherwise* $f(a) = 1_J$*, is a Kleene algebra homomorphism.*
(ii) The first injection $j\colon \mathbf{J} \to \mathbf{J} + \mathbf{K}$ *is one-to-one.*

Proof. Since $\mathbf{K}$ is non-trivial, we possibly have a Kleene algebra homomorphisms from $\mathbf{K}$ to the non-trivial one. For each $a, b \in K$, if $a \neq 0_K$ and $b \neq 0_K$, $a +_K b \neq 0_K$ and $a \cdot_K b \neq 0_K$ since $\mathbf{K}$ is integral. So, *(i)* follows from

$$\begin{aligned} &f(a) +_J f(b) = 1_J +_J 1_J = 1_J = f(a +_K b) \\ &f(a) \cdot_J f(b) = 1_J \cdot_J 1_J = 1_J = f(a \cdot_K b) \\ &f(a)^{*_J} = 1_J^{*_J} = 1_J = f(a^{*_K}) \ . \end{aligned}$$

(ii) will be proved using f given in *(i)*. Take $\mathrm{id}_{\mathbf{J}}$ and f, then a unique intermediating arrow $h\colon \mathbf{J} + \mathbf{K} \to \mathbf{J}$ with respect to them exists. By the definition of coproducts, h satisfies $\mathrm{id}_{\mathbf{J}} = h \circ j$. Thus j is one-to-one.

Set and **ISR** denote the categories of sets and functions, idempotent semirings and their homomorphisms, respectively. $U_K\colon$ **Kleene** $\to$ **Set** denotes the forgetful functor which takes a Kleene algebra to its carrier set. The functor U_K is decomposed by functors $U_{KI}\colon$ **Kleene** $\to$ **ISR** and $U_I\colon$ **ISR** $\to$ **Set**, where

$U_{KI}(\mathbf{K})$ is an idempotent semiring obtained by forgetting the * operator and U_I takes an idempotent semiring to its carrier set. These two functors U_{KI} and U_I have left adjoints F_{IK} and F_I respectively. $F_K \stackrel{\text{def}}{=} F_{IK} \circ F_I$ is a left adjoint to U_K.

Remark 2. For a set Σ, $\mathbf{Reg}(\Sigma)$ denotes the Kleene algebra consisting of the set of regular sets over Σ together with the standard operations on regular sets. Clearly, $\mathbf{Reg}(\Sigma)$ is integral. Moreover, it is known that $\mathbf{Reg}(\Sigma) \cong F_K(\Sigma)$.

The situation we state above is as follows:

$$\mathbf{Set} \underset{U_I}{\overset{F_I}{\rightleftarrows}} \mathbf{ISR} \underset{U_{KI}}{\overset{F_{IK}}{\rightleftarrows}} \mathbf{Kleene} \qquad (F_I \dashv U_I,\ F_{IK} \dashv U_{KI})$$

$$\begin{array}{ll} F_B \stackrel{\text{def}}{=} F_{IB} \circ F_I & F_K \stackrel{\text{def}}{=} F_{IK} \circ F_I \\ U_B = U_I \circ U_{BI} & U_K = U_I \circ U_{KI} \end{array}$$

For each idempotent semiring $\mathbf{S}$ and Kleene algebra $\mathbf{K}$, the bijection from $\mathbf{Kleene}(F_{IK}(\mathbf{S}), \mathbf{K})$ to $\mathbf{ISR}(\mathbf{S}, U_{KI}(\mathbf{K}))$, constituting the adjunction $F_{IK} \dashv U_{KI}$, is denoted by $\varphi_{\mathbf{S},\mathbf{K}}$. The subscript $\mathbf{S}, \mathbf{K}$ will be omitted unless confusions occur.

Let $\mathbf{S} = (S, +, \cdot, 0, 1)$ be an idempotent semiring. A nonempty subset $A \subseteq S$ closed under $+$ and downward under the ordering $\leq$ with respect to $+$ is called an ideal of $\mathbf{S}$. As the case of *-continuous Kleene algebras [2, 8], the set $\mathcal{I}_S$ of ideals forms a standard Kleene algebra ($\mathbf{S}$-algebra) [2] (or a quantale [13]). In an $\mathbf{S}$-algebra, the * operation can be defined as reflexive transitive closure, then, we have a Kleene algebra $\mathbf{K}_{\mathcal{I}_S}$. The mapping $\langle \_ \rangle$ which takes $a \in S$ to the least ideal which contains a, indeed, $\langle a \rangle = \{b \in S \mid b \leq a\}$, determines a one-to-one idempotent semiring homomorphism $\langle \_ \rangle \colon \mathbf{S} \to \mathbf{K}_{\mathcal{I}_S}$.

For an idempotent semiring $\mathbf{S}$, the arrow $\varphi(\mathrm{id}_{F_{IK}(\mathbf{S})})$ in $\mathbf{ISR}$ is a component of the unit of $F_{IK} \dashv U_{KI}$ with respect to $\mathbf{S}$. Take the arrow $\langle \_ \rangle \colon \mathbf{S} \to \mathbf{K}_{\mathcal{I}_S}$ in $\mathbf{ISR}$, then, by the universality of $F_{IK} \dashv U_{KI}$, we have the arrow $\overline{\langle \_ \rangle} \colon F_{IK}(\mathbf{S}) \to \mathbf{K}_{\mathcal{I}_S}$ in $\mathbf{Kleene}$ such that $\langle \_ \rangle = U_{KI}(\overline{\langle \_ \rangle}) \circ \varphi(\mathrm{id}_{F_{IK}(\mathbf{S})})$. Since $\langle \_ \rangle$ is one-to-one, $\varphi(\mathrm{id}_{F_{IK}(\mathbf{S})})$ is also one-to-one.

This fact induces the following property:

Proposition 2. *φ preserves one-to-one mapping.*

Proof. Let $m \colon F_{IK}(\mathbf{S}) \to \mathbf{K}$ be a one-to-one Kleene algebra homomorphism. Since $\varphi(\mathrm{id}_{F_{IK}(\mathbf{S})})$ is a component of the unit of $F_{IK} \dashv U_{KI}$ with respect to $\mathbf{S}$, we have $\varphi(m) = U_{KI}(m) \circ \varphi(\mathrm{id}_{F_{IK}(\mathbf{S})})$. Also, since $U_{KI}(m)$ is m itself and both of m and $\varphi(\mathrm{id}_{F_{IK}(\mathbf{S})})$ are one-to-one, $\varphi(m)$ is also one-to-one.

4 Framework

This section provides a framework for Kleene algebras with an embedded structure.

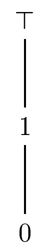
Let $\mathcal{X}$ be the category of a certain kind of algebras which have underlying idempotent semiring structure and homomorphisms. The forgetful functor from $\mathcal{X}$ to **ISR** will be denoted by $U_{XI}\colon\ \mathcal{X} \to \mathbf{ISR}$.

Definition 3 (Framework). A *framework* $\mathcal{F}_{-}$ for Kleene algebra with an embedded structure is the mapping which takes the category $\mathcal{X}$ of a certain kind of algebras which have underlying idempotent semiring structure and homomorphisms to the comma category $\langle U_{XI}, U_{KI}\rangle$.

Note that the third component of an object $\langle \mathbf{X}, \mathbf{K}, i\rangle$ in $\mathcal{F}_{\mathcal{X}}$ is not necessary inclusion.

Bool denotes the category of Boolean algebras and their homomorphisms. An object $\langle \mathbf{B}, \mathbf{K}, i\rangle$ in $\mathcal{F}_{\mathbf{Bool}}$ may not be a Kleene algebra with tests in the sense of [10, 11] but in the sense of [5]. It is known that the image of $\mathbf{B}$ under i forms Boolean algebra again. However, if we consider $\mathcal{F}_{\mathbf{BM}}$, where **BM** denotes the category of Boolean monoids and their homomorphisms, the third component of an object may not preserve structure of Boolean monoids.

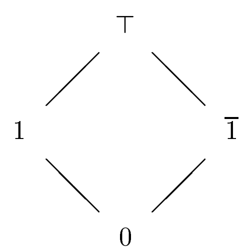
Example 1.



Consider the Kleene algebra $K = (\{0, 1, \top\}, +, \cdot, {}^*, 0, 1)$ with $+$ defined by the least upper bound with respect to the ordering, $\cdot$ defined by the table

$\cdot$	0	1	$\top$
0	0	0	0
1	0	1	$\top$
$\top$	0	$\top$	$\top$

and * defined by $0^* = 1^* = 1$ and $a^* = a$. This Kleene algebra appeared in [4].



Also consider the Boolean monoid $M = (\{0, 1, \overline{1}, \top\}, +, \sqcap, {}^-, \cdot, 0, \top, 1)$ with operators $+, \sqcap, {}^-$ defined by the least upper bound, the greatest lower bound, and the compliment with respect to the ordering, and $\cdot$ defined by the table

$\cdot$	0	1	$\overline{1}$	$\top$
0	0	0	0	0
1	0	1	$\overline{1}$	$\top$
$\overline{1}$	0	$\overline{1}$	$\overline{1}$	$\overline{1}$
$\top$	0	$\top$	$\overline{1}$	$\top$

Define the mapping $f\colon \{0,1,\top\} \to \{0,1,\overline{1},\top\}$ by

$$0 \mapsto 0, \quad 1 \mapsto 1, \quad \overline{1} \mapsto \top, \quad \top \mapsto \top \ .$$

Then f is an idempotent semiring homomorphism from $U_{MI}(M)$to $U_{KI}(K)$,where U_{MI} is the forgetful functor from **BM** to **ISR**. Thus, $\langle M, K, f\rangle$ is an object of $\mathcal{F}_{\mathbf{BM}}$ whose third component does not preserve Boolean monoid structure.

Let **RA** be the category of relation algebras and their homomorphisms. An object in $\mathcal{F}_{\mathbf{RA}}$ may not be a Kleene algebra with relations in the sense of [3].

Let $\langle \mathbf{X}, \mathbf{K}, i\rangle$ be an object of $\mathcal{F}_{\mathcal{X}}$. If i is one-to-one mapping, the image $\mathsf{im}(i)$ of the carrier set X of $\mathbf{X}$ forms an object $i[\mathbf{X}]$ in $\mathcal{X}$, and $\langle \mathbf{X}, \mathbf{K}, i\rangle$ is isomorphic to $\langle i[\mathbf{X}], \mathbf{K}, \subseteq\rangle$, where $i[\mathbf{X}]$ is exactly embedded in the Kleene algebra **K**.

For a category $\mathcal{X}$, define $\Psi_{\mathcal{X}}$ to be the functor from $\mathcal{F}_{\mathcal{X}}$ to $\mathcal{X} \times \mathbf{Kleene}$ which takes an object $\langle \mathbf{X}, \mathbf{K}, f\rangle$ to the pair $(\mathbf{X}, \mathbf{K})$ and an arrow $\langle h, k\rangle$ to the pair (h, k).

Also, for a category $\mathcal{X}$, define $\Phi_{\mathcal{X}}$ to be the functor from $\mathcal{X} \times \mathbf{Kleene}$ to $\mathcal{F}_{\mathcal{X}}$ which takes $(\mathbf{X}, \mathbf{K})$ to $\langle \mathbf{X}, F_{IK}(U_{XI}(\mathbf{X})) + \mathbf{K}, \varphi(i)\rangle$, where i is the first injection of the coproduct $F_{IK}(U_{XI}(\mathbf{X})) + \mathbf{K}$, and (f, g) to $\langle f, F_{IK}(U_{XI}(f)) + g\rangle$ in $\mathcal{F}_{\mathcal{X}}$.

The next property follows from Theorem 1.

Theorem 2. *$\Phi_{\mathcal{X}}$ is a left adjoint to $\Psi_{\mathcal{X}}$.*

Let U_X be the forgetful functor from $\mathcal{X}$ to **Set** which takes an object **X** to the carrier set X of **X** and an arrow $f\colon \mathbf{X} \to \mathbf{X}'$ to a function $f\colon X \to X'$ from the carrier set X of **X** to the carrier set X' of $\mathbf{X}'$. If U_X has a left adjoint $F_X\colon \mathbf{Set} \to \mathcal{X}$, the functor $U_X \times U_K\colon \mathcal{X} \times \mathbf{Kleene} \to \mathbf{Set} \times \mathbf{Set}$ has a left adjoint $F_X \times F_K$. In this case, we have the following sequence of adjunctions:

$$\mathbf{Set} \times \mathbf{Set} \underset{U_X \times U_K}{\overset{F_X \times F_K}{\rightleftarrows}} \mathcal{X} \times \mathbf{Kleene} \underset{\Psi_{\mathcal{X}}}{\overset{\Phi_{\mathcal{X}}}{\rightleftarrows}} \mathcal{F}_{\mathcal{X}}$$

5 Observation

$\mathcal{F}_{\mathcal{X}}^{\subseteq}$ denotes the subcategory of $\mathcal{F}_{\mathcal{X}}$ such that the third component of each object is inclusion. [5] showed that, for each object $\langle \mathbf{B}, \mathbf{K}, i\rangle$ in $\mathcal{F}_{\mathbf{Bool}}$, since the image $\mathsf{im}(i)$ of $U_{BI}(\mathbf{B})$ under the idempotent semiring homomorphism i forms Boolean algebra $i[\mathbf{B}]$, we have an object $\langle i[\mathbf{B}], \mathbf{K}, \subseteq\rangle$ in $\mathcal{F}^{\subseteq}_{\mathcal{X}}$. This fact induces the functor from $\mathcal{F}_{\mathbf{Bool}}$ to $\mathcal{F}^{\subseteq}_{\mathbf{Bool}}$, and, moreover, the functor is the left adjoint to the forgetful functor from $\mathcal{F}^{\subseteq}_{\mathbf{Bool}}$ to $\mathcal{F}_{\mathbf{Bool}}$.

On the other hand, Example 1 shows that the third component f of $\langle B, K, f\rangle$ does not preserve Boolean monoid structure. So, we cannot have adjunction between $\mathcal{F}_{\mathbf{BM}}$ and $\mathcal{F}^{\subseteq}_{\mathbf{BM}}$ in the similar way to the case of $\mathcal{F}_{\mathbf{Bool}}$ and $\mathcal{F}^{\subseteq}_{\mathbf{Bool}}$. Though we may not have adjunction between $\mathcal{F}^{\subseteq}_{\mathcal{X}}$ and $\mathcal{F}_{\mathcal{X}}$ for each $\mathcal{X}$, the adjunction between $\mathbf{Set} \times \mathbf{Set}$ and $\mathcal{F}^{\subseteq}_{\mathcal{X}}$ still can be obtained.

Assume that $U_X\colon\ \mathcal{X} \to \mathbf{Set}$ has a left adjoint $F_X\colon\ \mathbf{Set} \to \mathcal{X}$. For a set A, the carrier of $F_X(A)$ should have at least two distinct elements which may be the zero element and the identity element in $U_{XI}(F_X(A))$ because $F_X(A)$ is a free algebra which has underlying idempotent semiring structure. So, $F_{IK}(U_{XI}(F_X(A)))$ is a non-trivial Kleene algebra since the unit of $F_{IK} \dashv U_{KI}$ is one-to-one. Also, by Remark 2, $F_K(\Sigma)$ is integral. Thus we have the following property.

Theorem 3. *$\varphi(i)$ of an object $\langle F_X(A), F_{IK}(U_{XI}(F_X(A)))+F_K(B), \varphi(i)\rangle$ in $\mathcal{F}_{\mathcal{X}}$ is one-to-one for each pair (A, B) of sets.*

Proof. Take the object $\langle F_X(A), F_{IK}(U_{XI}(F_X(A))) + F_K(B), \varphi(i)\rangle$ in $\mathcal{F}_{\mathcal{X}}$. Then i is one-to-one by Proposition 1. Therefore, by Proposition 2, $\varphi(i)$ is one-to-one.

Therefore, as we mentioned in Section 4, $\mathsf{im}(\varphi(i))$ forms an object $\varphi(i)[F_X(A)]$ in $\mathcal{X}$, and

$$\begin{aligned}&\langle F_X(A), F_{IK}(U_{XI}(F_X(A))) + F_K(B), \varphi(i)\rangle \\ &\quad\cong \langle \varphi(i)[F_X(A)], F_{IK}(U_{XI}(F_X(A))) + F_K(B), \subseteq\rangle\ .\end{aligned}$$

The mapping $(A, B) \mapsto \langle \varphi(i)[F_X(A)], F_{IK}(U_{XI}(F_X(A))) + F_K(B), \subseteq\rangle$ determines a functor F from $\mathbf{Set} \times \mathbf{Set}$ to $\mathcal{F}^{\subseteq}_{\mathcal{X}}$. Then the following holds:

Theorem 4. *The functor F is a left adjoint to the forgetful functor from $\mathcal{F}^{\subseteq}_{\mathcal{X}}$ to* $\mathbf{Set} \times \mathbf{Set}$.

The following is immediate since $U_B\colon\ \mathbf{Bool} \to \mathbf{Set}$ has a left adjoint F_B.

Corollary 1. *$\langle \varphi(i)[F_B(A)], F_{IK}(U_{BI}(F_B(A))) + F_K(B), \subseteq\rangle$ is the free Kleene algebra with tests, in the sense of [10, 11], generated by a pair of sets A and B.*

Since the structure of relation algebras is obtained by adding the composition, the identity element with respect to the composition, and the converse to Boolean algebra structure and axiomatised by equations, the forgetful functor $U_R\colon\ \mathbf{RA} \to \mathbf{Set}$ also has a left adjoint $F_R\colon\ \mathbf{Set} \to \mathbf{RA}$. Note that the statement is not true if we require relation algebras are complete.

Corollary 2. *$\langle \varphi(i)[F_R(A)], F_{IK}(U_{RI}(F_R(A))) + F_K(B), \subseteq\rangle$ is the free Kleene algebra with relations, in the sense of [3], generated by a pair of sets A and B if we do not require the completeness of relation algebras.*

6 Conclusion

We proposed general framework for a Kleene algebra with an embedded structure. If the embedded structure satisfies the following conditions we have a free construction from a pair of sets.

- $\mathcal{X}$ is the category of a certain kind of algebras which have underlying idempotent semiring structure and homomorphisms,
- $U_X : \mathcal{X} \to \mathbf{Set}$ has a left adjoint.

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