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Sequential continuity and boundedness of generalized functions in constructive mathematics (Preliminary Version)

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概要

Sequential continuity and boundedness of generalised functions are equivalent in classical mathematics. In this paper we show that the equivalency is also provable in Brouwer's intuitionistic mathematics and constructive recursive mathematics of Markov's school but is not in Bishop's constructive mathematics. This fact is an important remark to develop generalized function theory just in Bishop's framework.

To this end, we investigate structure of a generalized function whose support consists of the origin.

1 Introduction

We can prove that the following three propositions are equivalent in Bishop's constructive mathematics (BISH): a linear mapping of a separable Banach space into a normed space is sequentially continuous if and only if it is bounded; the Banach-Steinhaus theorem (for a sequence $\{u_n\}$ of bounded linear mappings of a separable Banach space X into a normed space Y, if $u(x) \equiv \lim_n u_n(x)$ exists for all x in X, then the limit u is bounded); the principle BD-$\mathbb{N}$. Here BD-$\mathbb{N}$ is provable in classical mathematics, Brouwer's intuitionistic mathematics and constructive recursive mathematics of Markov's school, but is not in BISH; see Section 2 for further details. It is easy to prove by [5, Theorem 4] and [2, ch. 7, Proposition 1.5] that the first proposition follows from BD-$\mathbb{N}$. The Banach-Steinhaus theorem for sequentially continuous linear mappings of a separable Banach space into a normed space is provable in BISH (see [6, Theorem 7]), and therefore the first proposition implies the second one. The Banach-Steinhaus theorem for bounded ones implies BD-$\mathbb{N}$, from [7, Theorem 21]. We then see that, in BISH, the space of bounded ones is included in the space of sequentially continuous ones, and that it cannot be provable that the two spaces are equal. Also, the space of sequentially continuous ones is closed under convergence with respect to weak topology, but it cannot be provable that the space of bounded ones is.

These matters can be showed for other spaces. The space $\mathcal{E}(\mathbb{R})$ consists of infinitely differentiable functions on $\mathbb{R}$, and is a complete space with a metrizable locally convex structure; see Section 2 more details. The Banach-Steinhaus theorem for sequentially continuous linear functionals on $\mathcal{E}(\mathbb{R})$ is provable in BISH (see [12, Corollary 3.1]). On the other hand, the following propositions are equivalent in BISH: a linear functional on $\mathcal{E}(\mathbb{R})$ is sequentially continuous if and only if it is bounded; the Banach-Steinhaus theorem for bounded ones; the principle BD-$\mathbb{N}$. In this paper these matters are moreover showed for the space $\mathcal{D}(\mathbb{R})$, which is the non-metrizable locally convex space of test functions (i.e. infinitely differentiable functions on $\mathbb{R}$ with compact support). This space is complete if and only if BD-$\mathbb{N}$ can be proved (see [8, Theorem 4] and [11, Corollary 3.5]). In particular, a continuous linear functional on $\mathcal{D}(\mathbb{R})$ is called a *distribution* or a *generalized function*. That is, the results of this paper give an important remark to develop generalized function theory just in BISH. It will be actually showed that it is not provable in BISH that the limit of a sequence of bounded distributions is a bounded one, although the limit of a sequence of sequentially continuous ones is a sequentially continuous one in BISH.

The aim of this paper is to prove that the foregoing three propositions for the space $\mathcal{D}(\mathbb{R})$ are equivalent in BISH. To this end, we consider structure of a distribution whose support consists of the origin, and prove two constructive versions of representation theorem for such a distribution in Section 3. Section 2 gives preliminaries on locally convex spaces with seminorms, constructive calculus and so on. The main theorem of this paper is showed in Section 4. We also give notes with respect to the main result for more general locally convex spaces in Section 5.

2 Preliminaries

Let X be a vector space over $\mathbb{R}$. A function $p : X \to \mathbb{R}^{0+}$ is said to be a *seminorm* on X if it satisfies that for $x, y \in X$ and $\lambda \in \mathbb{R}$, (1) $p(x+y) \leq p(x) + p(y)$ and (2) $p(\lambda x) = |\lambda| p(x)$. Let I be a set, and $\{p_i\}_{i\in I}$ seminorms on X. A pair $(X, \{p_i\}_{i\in I})$ is said to be a *locally convex space* over $\mathbb{R}$ if for each $x \in X$, whenever $p_i(x) = 0$ for all $i \in I$, then $x = 0$.

Let $(X, \{p_i\}_{i\in I})$ be a locally convex space over $\mathbb{R}$. A sequence $\{x_n\}$ in X is said to *converges* in X if for each k and $i_1, \ldots, i_l \in I$, there exists N such that $\max_{1\leq m\leq l} p_{i_m}(x_n - x) < 2^{-k}$ for all $n \geq N$. A sequence $\{x_n\}$ is a *Cauchy sequence* in X if for each k and $i_1, \ldots, i_l \in I$, there exists N such that $\max_{1\leq t\leq l} p_{i_t}(x_m - x_n) < 2^{-k}$ for all m and n with $m, n \geq N$. A locally convex space X is said to be *complete* if every Cauchy sequence converges in X.

A vector space is said to be a $(\mathcal{F})$*-space* if it is a complete locally convex space with countably many seminorms. Note that a $(\mathcal{F})$-space is a complete metric space, and is a filter-complete metric space; see [11, Section 1].

Let $(X, \{p_i\})$ be a locally convex space, and $\{(X_k, \{p^n_m\}_m)\}_k$ a sequence of $(\mathcal{F})$-spaces such that

- $X = \bigcup_{k=0}^{\infty} X_k$,
- for each n, X_k is a proper subset of X_{k+1},
- for each k, the topology of X_k is equivalent to one induced from X_{k+1}; that is,

$$\forall k \forall m \exists k' \exists m' \forall x \in X_k [p^k_{m'}(x) < 2^{-k'} \Rightarrow p^{k+1}_m(x) < 2^{-k}],$$
$$\forall k' \forall m' \exists k \exists m \forall x \in X_k [p^{k+1}_m(x) < 2^{-k} \Rightarrow p^k_{m'}(x) < 2^{-k'}].$$

Set

$$V_{i_1,\ldots,i_l,k} := \{x \in X : \max_{1\leq m\leq l} p_{i_m}(x) < 2^{-k}\} \quad (i_1, \ldots, i_l \in I, k \in \mathbb{N}),$$

and let $\mathcal{B}_0$ be the class of all $V_{i_1,\ldots,i_l,k}$. $\mathcal{B}'_0$ denotes the class of all subsets V of X, satisfying the following four conditions:

1. for each s and t in $\mathbb{R}$, if $s, t \geq 0 \wedge s + t = 1$, then $sV + tV \subset V$ (*convexity*),
2. for each r in $\mathbb{R}$, if $|r| \leq 1$ then $rV \subset V$ (*circledness*),
3. for each x in X, there exists $a > 0$ such that $|r| \leq a$ implies $rx \in V$ for all $r \in \mathbb{R}$ (*absorption*) .
4. for each k, the intersection $V \cap X_k$ is a neighbourhood of 0 in X_k; that is, there exist m and k in $\mathbb{N}$ such that for all x in X_k, if $p^k_m(x) < 2^{-k}$ then $x \in V$.

We then say that $(X, \{p_i\}_{i\in I})$ is a $(\mathcal{LF})$*-space*, and that $\{(X_k, \{p^k_m\}_m)\}_k$ is a *sequence of definition of* X, if $\mathcal{B}_0$ is equivalent to $\mathcal{B}'_0$; that is,

$$\forall V \in \mathcal{B}_0 \exists V' \in \mathcal{B}'_0 (V' \subset V) \quad \text{and} \quad \forall V' \in \mathcal{B}'_0 \exists V \in \mathcal{B}_0 (V \subset V').$$

Note that if $(X, \{p_i\})$ is a $(\mathcal{LF})$-space with a sequence $\{(X_k, \{p^k_m\}_m)\}_k$ of separable $(\mathcal{F})$-spaces, then X is separable. In fact, the union of countable dense subsets of all X_k is a countable dense subset of X.

A *functional* on a locally convex space X is a function from X to $\mathbb{R}$. A functional u is said to be *sequentially continuous* on a locally convex space X if for each sequence $\{x_n\}$ in X and $x \in X$, if $\{x_n\}$ converges to x in X, then the sequence $\{u(x_n)\}$ converges to $u(x)$ in $\mathbb{R}$. We can prove the Banach-Steinhaus theorem for $(\mathcal{F})$-spaces, as given in [12, Thorem 2.3]: for a sequence $\{u_n\}$ of sequentially continuous linear functionals on a $(\mathcal{F})$-space X, if $u(x) := \lim_n u_n(x)$ exists for all x in X, then u is a sequentially continuous linear functional on X. A linear functional u on a locally convex space X is said to be *bounded* if there exist $C > 0$ and $i_1, \dots, i_l \in I$ such that

$$|u(x)| \leq C \max_{1 \leq m \leq l} p_{i_m}(x) \quad (x \in X).$$

Ii is clear that every bounded linear functional on a locally convex space is sequentially continuous. Note that, given a linear functional u on a $(\mathcal{LF})$-space X with a sequence $\{X_k\}$ of $(\mathcal{F})$-spaces, u is bounded if and only if it is bounded on each X_k; see [12, Prposition 4.2].

We say that a subset A of $\mathbb{N}$ is *pseudobounded* if for any sequence $\{a_n\}$ in A, $a_n < n$ for all sufficiently large n. Note that A is pseudobounded if and only if for any sequence $\{a_n\}$ in A, $\lim_{n\to\infty} \frac{a_n}{n} = 0$ (see [8, Lemma 3]). Any bounded subset of $\mathbb{N}$ is pseudobounded. On the other hand, the converse

BD Every nonempty pseudobounded subset of $\mathbb{N}$ is bounded

cannot be proved in BISH. In fact, a natural recursivisation of the following principle is independent of Heyting arithmetic (see [4]):

BD-$\mathbb{N}$ Every countable pseudobounded subset of $\mathbb{N}$ is bounded.

In the rest of this paper we assume familiarity with constructive calculus, as found in [1, Chapter 2], [3, Appendix], [2, Chapter 2] or [9, Chapter 6].

Let f and f' be functions from $\mathbb{R}$ to itself, and X a subset of $\mathbb{R}$. f is said to be *uniformly continuous* on X if for each k, there exists N such that for all x and y in X,

$$|x - y| < 2^{-N} \implies |f(x) - f(y)| < 2^{-k}.$$

We say that f is *continuous* on $\mathbb{R}$ if it is uniformly continuous on each compact interval. Also, f is said to be *uniformly differentiable* on X with *derivative* f' if for each k, there exists N such that for all x and y in X,

$$|x - y| < 2^{-N} \implies |f'(x)(x - y) - (f(x) - f(y))| \leq 2^{-k}|x - y|.$$

We say that f is *differentiable* on $\mathbb{R}$ with *derivative* f' if f is uniformly differentiable with derivative f' for uniform differentiation on each compact interval. It is clear from [3, Appendix A] that a differentiable function on $\mathbb{R}$ and its derivative are continuous on $\mathbb{R}$. We use the familiar notation for iterated derivatives of differentiability: $f^{(0)} := f$, $f^{(l+1)} := (f^{(l)})'$. A function $f : \mathbb{R} \to \mathbb{R}$ is said to be *infinitely differentiable* on $\mathbb{R}$ if for each l, there exists the l-th derivative $f^{(l)}$ of f for differentiation on $\mathbb{R}$.

Let $\mathcal{E}(\mathbb{R})$ be the locally convex space of infinitely differentiable functions from $\mathbb{R}$ to $\mathbb{R}$, with the seminorms

$$\|f\|_n := \max_{l \leq n} \sup_{|x| \leq n} |f^{(l)}(x)| \quad (f \in \mathcal{E}(\mathbb{R}), n \in \mathbb{N}).$$

$\mathcal{E}(\mathbb{R})$ is a separable ($\mathcal{F}$)-space; see [11, Sections 1 and 2].

For a function $f : \mathbb{R} \to \mathbb{R}$, let supp f be the closure of the set $\{x \in \mathbb{R} | |f(x)| > 0\}$ in $\mathbb{R}$. f is said to *have compact support* if the set supp f is bounded. We call an infinitely differentiable function on $\mathbb{R}$ with compact support a *test function*. It is easy to show that if ϕ and ψ are test functions, then so is the multiplication $\phi\psi$.

Given any k in $\mathbb{N}$, let $\mathcal{D}_k(\mathbb{R})$ denote the space of test functions ϕ such that the set supp ϕ are contained in $[-k, k]$, with the seminorms

$$\|\phi\|_{k,m} := \max_{l \leq m} \sup_{|x| \leq 1} |\phi^{(m)}(x)| \quad (m \in \mathbb{N}, \phi \in \mathcal{D}_k(\mathbb{R})).$$

Note that every $\mathcal{D}_k(\mathbb{N})$ is a separable ($\mathcal{F}$)-space (see [12, Proposition 4.5]).

Let $\mathcal{D}(\mathbb{R})$ denote the space of test functions, with the seminorms

$$p_{\alpha,\beta}(\phi) := \sup_{n} \max_{l \leq \beta(n)} \sup_{|x| \geq n} 2^{\alpha(n)} |\phi^{(l)}(x)| \quad (\phi \in \mathcal{D}(\mathbb{R}), \alpha, \beta \in \mathbb{N} \to \mathbb{N}).$$

This space is complete if and only if BD-$\mathbb{N}$ can be proved, see [8, Theorem 4] and [11, Corollary 3.5]. Note that $\mathcal{D}(\mathbb{R})$ is a ($\mathcal{LF}$)-space with the sequence $\{(\mathcal{D}_k(\mathbb{R}), \{\| \cdot \|_{k,m}\})\}$; see [12, Theorem 4.1] for more details.

In the rest of this paper, a *distribution* or *generalized function* is a sequentially continuous linear functional on $\mathcal{D}(\mathbb{R})$. Write $\langle u, \phi \rangle := u(\phi)$ for each linear functional u and element ϕ in $\mathcal{D}(\mathbb{R})$ or $\mathcal{E}(\mathbb{R})$. The Banach-Steinhaus theorem for distributions can be proved, as showed in [10, Theorem 4.10].

Let u be a linear functional on $\mathcal{D}(\mathbb{R})$. Set

$$\begin{aligned} \text{supp } u &:= \{x \in \mathbb{R} : \forall \varepsilon > 0 \exists \phi \in \mathcal{D}(\mathbb{R}) [\text{supp } \phi \subset (x - \varepsilon, x + \varepsilon) \wedge |\langle u, \phi \rangle| > 0]\}, \\ \text{supp}_{\mathbb{N}} u &:= \{0\} \cup \{n \in \mathbb{N} : \exists x \in \mathbb{R} [|x| > n \wedge x \in \text{supp } u]\}. \end{aligned}$$

Note that for any ϕ in $\mathcal{D}(\mathbb{R})$, if $|\langle u, \phi \rangle| > 0$, then we have supp $u \cap$ supp $\phi \neq \emptyset$; see [12, Proposition 5.1]. This fact implies that if supp $u = \{0\}$ and if there exists i such that $\phi(x) = 0$ for all x with $|x| < 2^{-i}$, then $\langle u, \phi \rangle = 0$. We say that u *has compact support* if supp u is bounded, and do that u *has pseudobounded support* if $\text{supp}_{\mathbb{N}}$ u is pseudobounded. A distribution with pseudobounded support can be uniquely extend to the space $\mathcal{E}(\mathbb{R})$ (see [12, Theorem 6.2]). Conversely, a sequentially continuous linear functional on $\mathcal{E}(\mathbb{R})$ has pseudobounded support (see [12, Theorem 6.3]). A bounded distribution with compact support can be uniquely extend to the space $\mathcal{E}(\mathbb{R})$. Conversely, a bounded linear functional on $\mathcal{E}(\mathbb{R})$ has compact support. On the other hand, the following two properties are equivalent to BD-$\mathbb{N}$: every bounded distribution with pseudobounded support is bounded on $\mathcal{E}(\mathbb{R})$; every sequentially continuous linear functional on $\mathcal{E}(\mathbb{R})$ is a distribution with compact support (see [12, Theorem 7.3]).

3 Structure of a distribution whose support consists of the origin

We in this section consider distributions whose support consists of the origin; that is, the support is the singleton set $\{0\}$. Such a distribution is uniquely extended to $\mathcal{E}(\mathbb{R})$; see the last paragraph of Section 2.

Lemma 1. *Let u be a distribution, k in $\mathbb{N}$. Then the set*

$$A \equiv \{0\} \cup \{n \in \mathbb{N} : \exists C, L \in \mathbb{N} \exists \phi \in \mathcal{D}_k(\mathbb{R}) \, [L, C \geq n \wedge |\langle u, \phi \rangle| > C \|\phi\|_{k,L}]\}$$

is pseudobounded.

Proof. Let $\{a_n\}$ be any sequence in A. Construct a binary sequence $\{\lambda_n\}$ such that $\lambda_0 = 0$ and

$$\lambda_n = 0 \implies a_n < n,$$
$$\lambda_n = 1 \implies a_n \geq n.$$

Define a sequence $\{\psi_n\}$ in $\mathcal{D}_k(\mathbb{R})$ as follows. If $\lambda_n = 0$, then set $\psi_n := 0$. If $\lambda_n = 1$, then there exist $\phi \in \mathcal{D}_k(\mathbb{R})$ and L and C in $\mathbb{N}$ such that $L, C \geq a_n \geq n$ and $|\langle u, \phi \rangle| > C \|\phi\|_{k,L}$, and then set $\psi_n := n \|\phi\|_{k,L}$. Note that $\{\psi_n\}$ is well-defined, since $|\langle u, \psi_n \rangle| > 0$ implies that $\|\psi_n\|_{k,L} > 0$ for some L in $\mathbb{N}$ by [12, Propistion 5.1]; see the last paragraph of Section 2. Then the sequence $\{\psi_n\}$ converges to 0 in $\mathcal{D}_k(\mathbb{R})$ as follows: for each l, if $n \geq l$, then for some $\phi \in \mathcal{D}_k(\mathbb{R})$ and $L \geq l$, we have

$$\|\psi_n\|_{k,l} \leq \frac{\|\phi\|_{k,l}}{n \|\phi\|_{k,L}} \leq \frac{1}{n}.$$

We now choose N such that $|\langle u, \psi_n \rangle| < 1$ for all $n \geq N$. If there exists $n \geq N$ such that $\lambda_n = 1$, then for some $\phi \in \mathcal{D}_k(\mathbb{R})$ and L and C in $\mathbb{N}$ with $L, C \geq a_n \geq n$,

$$1 > |\langle u, \psi_n \rangle| = \frac{|\langle u, \phi \rangle|}{n \|\phi\|_{k,L}} \geq \frac{|\langle u, \phi \rangle|}{C \|\phi\|_{k,L}} > 1,$$

a contradiction. Hence $\lambda_n = 0$ for all $n \geq N$. □

We can here construct the test functions θ_0 such that

- $0 \leq \theta_0(x) \leq 1 \quad (x \in \mathbb{R})$,
- $\theta_0(x) = 1 \quad \big(|x| \leq 2^{-1}\big)$ and
- $\theta_0(x) = 0 \quad (|x| \geq 1)$;

see [10, Lemma 4.3] for further details. Setting $\theta_m(x) := \theta_0(2^m x)$ for each m in $\mathbb{N}$, every θ_m satisfies that

- $\theta_m(x) = 1 \quad (|x| \leq 2^{-(m+1)})$ and
- $\theta_m(x) = 0 \quad (|x| \geq 2^{-m})$.

Note that for each m and n in $\mathbb{N}$,

$$\max_{l \leq n} \sup_{|x| \leq 1} \left| \theta_0^{(l)}(x) \right| = \max_{l \leq n} \sup_{|x| \leq 2^{-m}} \left| 2^{-lm} \theta_m^{(l)}(x) \right| \geq 2^{-nm} \max_{l \leq n} \sup_{|x| \leq 2^{-m}} \left| \theta_m^{(l)}(x) \right|.$$

Lemma 2. *Let u be a distribution whose support consists of the origin, and ϕ in $\mathcal{E}(\mathbb{R})$. Then for any m in $\mathbb{N}$,*

$$\langle u, \phi \rangle = \langle u, \theta_m \phi \rangle.$$

Proof. Fix any m in $\mathbb{N}$. If $|x| \leq 2^{-(m+1)}$, then we have $(1 - \theta_m(x))\phi(x) = 0$. Thus $|\langle u, \phi \rangle - \langle u, \theta_m \phi \rangle| = \langle u, (1 - \theta_m)\phi \rangle = 0$ by [12, Cororollary 5.2] (see the last paragraph of Section 2). □

The space $\mathcal{E}(\mathbb{R})$ is here separable. This fact is showed in [12, Lemma 3.2], by actually proving the following lemma:

Lemma 3. *For each m and n in $\mathbb{N}$ and infinitely differentiable function ϕ on $\mathbb{R}$, there exist $a_0, \cdots, a_k$ in $\mathbb{R}$ such that*

$$\max_{l\leq m} \sup_{|x|\leq m} \left| \phi^{(l)}(x) - \left(\sum_{i=0}^{k} \frac{a_i}{(i+m)!} x^{i+m} + \sum_{i=0}^{m} \frac{\phi^{(i)}(0)}{i!} x^i \right)^{(l)} \right| < 2^{-n}.$$

Proof. See the proof of [12, Lemma 3.2]. □

Let u be a distribution with pseudobounded support i.e. a sequentially continuous linear functional on $\mathcal{E}(\mathbb{R})$, and set $f_n(x) := x^n$ for arbitrary n in $\mathbb{N}$ and x in $\mathbb{R}$. We then write $\langle u, x^n \rangle := \langle u, f_n \rangle$.

We now give a constructive representation of a distribution whose support consists of the origin.

Theorem 4. *Let u be a distribution whose support consists of the origin. Then for each k in $\mathbb{N}$ and ϕ in $\mathcal{E}(\mathbb{R})$, there exists N in $\mathbb{N}$ such that*

$$\left| \langle u, \phi \rangle - \sum_{i=0}^{n} \frac{\langle u, x^i \rangle}{i!} \phi^{(i)}(0) \right| < 2^{-k} \quad (n \geq N).$$

Proof. Let k be in $\mathbb{N}$, and ϕ in $\mathcal{E}(\mathbb{R})$. Construct sequences $\{l_n\}$ in $\mathbb{N}$ and $\{a_i^n\}_{i\leq l_n, n}$ in $\mathbb{R}$ such that for all n,

$$\max_{l\leq n} \sup_{|x|\leq n} \left| \phi^{(l)}(x) - \left(\sum_{i=0}^{l_{n+1}} \frac{a_i^{n+1}}{(i+(n+1))!} x^{i+(n+1)} + \sum_{i=0}^{n+1} \frac{\phi^{(i)}(0)}{i!} x^i \right)^{(l)} \right| < 2^{-n},$$

from Lemma 3, and set

$$p_n(x) := \sum_{i=0}^{l_{n+1}} \frac{a_i^{n+1}}{(i+(n+1))!} x^{i+(n+1)} + \sum_{i=0}^{n+1} \frac{\phi^{(i)}(0)}{i!} x^i \quad (n \in \mathbb{N}).$$

Then the sequence $\{p_n\}$ converges to ϕ in $\mathcal{E}(\mathbb{R})$ by Lemma 3, and therefore the sequence $\{\langle u, p_n \rangle\}$ converges to $\langle u, \phi \rangle$ in $\mathbb{R}$ by sequential continuity on u. That is, there exists N_0 such that

$$|\langle u, \phi - p_n \rangle| < 2^{-(k+1)} \quad (n \geq N_0).$$

We now set

$$\begin{aligned} \psi_n(x) &:= p_n(x) - \sum_{i=0}^{n} \frac{\phi^{(i)}}{i!} x^{n+1} \\ &= \sum_{i=0}^{l_{n+1}} \frac{a_i^{(n+1)}}{(i+(n+1))!} x^{i+(n+1)} + \frac{\phi^{(n+1)}(0)}{(n+1)!} x^{n+1} \\ &= \left(\sum_{i=0}^{l_{n+1}} \frac{a_i^{(n+1)}}{(i+(n+1))!} x^i + \frac{\phi^{(n+1)}(0)}{(n+1)!} \right) x^{n+1} \quad (n \in \mathbb{N}), \end{aligned}$$

and $c := \max_{i\leq l_{n+1}} \frac{|a_i^{n+1}|}{(n+1-l_{n+1})!} \cdot l_{n+1} + \cdot \frac{\phi^{(n+1)}(0)}{(n+1)!}$. We then have

$$\max_{|x|\leq 2^{-m}} |\psi_n(x)| \leq \max_{|x|\leq 2^{-m}} \left| c x^{n+1} \right|.$$

Letting

$$A := \{0\}\cup \left\{ m \in \mathbb{N} : \exists C, L \in \mathbb{N} \exists \zeta \in \mathcal{D}_1(\mathbb{R}) \left[C, L \geq m \wedge |\langle u, \zeta\rangle| > C \max_{l\leq L} \sup_{|x|\leq 1} \left|\zeta^{(l)}(x)\right| \right] \right\},$$

the set A is pseudobounded by Lemma 1. Construct a binary sequence $\{\lambda_n\}$ such that

$$\lambda_n = 0 \Longrightarrow |\langle u, \psi_n\rangle| < 2^{-(k+1)}$$
$$,\lambda_n = 1 \Longrightarrow |\langle u, \psi_n\rangle| > 0.$$

Define a sequence $\{a_n\}$ in A as follows. If $\lambda_n = 0$, then we set $b_n := 0$. If $\lambda_n = 1$, then, by Lemma 2, we have

$$\langle u, \theta_m \psi_n\rangle = \langle u, \psi_n\rangle > 0$$

for all m in $\mathbb{N}$. We can moreover choose M in $\mathbb{N}$ satisfying the following conditions:

$$\begin{aligned}
&|\langle u, \theta_M \psi_n\rangle| \\
&> n \sum_{j=0}^{n} \binom{n}{j} \cdot \max_{l\leq n} \sup_{|x|\leq 1} \left|\theta_0^{(l)}(x)\right| \cdot c \cdot 2^{-M} \\
&= n \sum_{j=0}^{n} \binom{n}{j} \cdot \max_{l\leq n} \sup_{|x|\leq 2^{-M}} \left|2^{-Ml}\theta_M^{(l)}(x)\right| \cdot c \cdot 2^{-M} \\
&\geq n \sum_{j=0}^{n} \binom{n}{j} \cdot \max_{l\leq n} \sup_{|x|\leq 2^{-M}} \left|\theta_M^{(l)}(x)\right| \cdot c \cdot 2^{-M(n+1)}
\end{aligned}$$

$$\begin{aligned}
&\geq n \sum_{j=0}^{n} \binom{n}{j} \cdot \max_{l\leq n} \sup_{|x|\leq 2^{-M}} \left|\theta_M^{(l)}(x)\right| \cdot \max_{l\leq n} \sup_{|x|\leq 2^{-M}} \left|cx^{n+1}\right| \\
&\geq n \sum_{j=0}^{n} \binom{n}{j} \max_{l\leq n} \sup_{|x|\leq 2^{-M}} \left|\theta_M^{(l)}(x)\right| \cdot \max_{l\leq n} \sup_{|x|\leq 2^{-M}} \left|\psi_n^{(l)}(x)\right| \\
&\geq n \max_{l\leq n} \sup_{|x|\leq 2^{-M}} \left| \sum_{j=0}^{l} \binom{l}{j} \theta_M^{(l-j)}(x) \psi_n^{(j)}(x) \right| \\
&= n \max_{l\leq n} \sup_{|x|\leq 2^{-M}} \left|(\theta_M \psi_n)^{(l)}\right| \\
&= n \max_{l\leq n} \sup_{|x|\leq 1} \left|(\theta_M \psi_n)^{(l)}\right|,
\end{aligned}$$

and set $b_n := n$. Then $\{b_n\}$ is in A, and therefore there exists N_1 in $\mathbb{N}$ such that $b_n < n$ for $n \geq N_1$. If $\lambda_{n+1} = 0$ for some $n \geq N_1$, then we have $n > b_n = n$, a contradiction. Thus $\lambda_n = 1$ for $n \geq N_1$. We hence conclude that, for all $n \geq \max\{N_0, N_1\}$,

$$\begin{aligned}
\left|\langle u, \phi\rangle - \sum_{i=0}^{n} \frac{\langle u, x^i\rangle}{i!} \phi^{(i)}(0)\right| &= \left|\left\langle u, \phi - \sum_{i=0}^{n} \frac{\phi^{(i)}(0)}{i!} x^i \right\rangle\right| \\
&\leq |\langle u, \psi_n\rangle| + |\langle u, \phi - p_n\rangle| \\
&< 2 \times 2^{-(k+1)} = 2^{-k}.
\end{aligned}$$

□

Theorem 4 means that, for each test function ϕ, we can approximate the series $\left\{\sum_{i=0}^{n} \frac{\langle u,x^i\rangle}{i!}\phi^{(i)}(0)\right\}$ to $\langle u, \phi\rangle$. On the other hand, it will be showed in Theorem 10 that u can be approximated uniformly by $\left\{\sum_{i=0}^{n} \frac{\langle u,x^i\rangle}{i!}\phi^{(i)}(0)\right\}$ if and only if BD-$\mathbb{N}$ can be proved.

We next show a representation theorem for a bounded distribution whose support consists of the origin.

Theorem 5. *Let u be a bounded distribution whose support consists of the origin. Then there exists N in $\mathbb{N}$ such that*

$$\langle u, \phi\rangle = \sum_{i=0}^{N} \frac{\langle u, x^i\rangle}{i!}\phi^{(i)}(0) \quad (\phi \in \mathcal{D}(\mathbb{R}))\,.$$

Proof. We show that if there exist N in $\mathbb{N}$ and $C > 0$ such that $|\langle u,\phi\rangle| \le C \max_{l\le N}\sup_{|x|\le 1}\left|\phi^{(l)}(x)\right|$ for all ϕ in $\mathcal{D}(\mathbb{R})$, then we obtain that $\langle u,\phi\rangle = \sum_{i=0}^{N}\frac{\langle u,x^i\rangle}{i!}\phi^{(i)}(0)$ for all ϕ in $\mathcal{D}(\mathbb{R})$. Assume that there exists ϕ in $\mathcal{D}(\mathbb{R})$ such that $\left|\langle u,\phi\rangle - \sum_{i=0}^{N}\frac{\langle u,x^i\rangle}{i!}\phi^{(l)}(0)\right| > 0$. By boundedness on u and Theorem 4, there exists N_1 in $\mathbb{N}$ such that $N_1 > N$ and

$$\left|\langle u,\phi\rangle - \sum_{i=0}^{N_1}\frac{\langle u,x^i\rangle}{i!}\phi^{(i)}(0)\right| < \left|\langle u,\phi\rangle - \sum_{i=0}^{N}\frac{\langle u,x^i\rangle}{i!}\phi^{(l)}(0)\right|;$$

that is,

$$\left\langle u, \sum_{i=N+1}^{N_1}\frac{\phi^{(i)}(0)}{i!}x^i\right\rangle > 0.$$

We can then choose M in $\mathbb{N}$ such that

$$\begin{aligned}
&\left|\left\langle u, \theta_M \sum_{i=N+1}^{N_1}\frac{\phi^{(i)}(0)}{i!}x^i\right\rangle\right| = \left|\left\langle u, \sum_{i=N+1}^{N_1}\frac{\phi^{(i)}(0)}{i!}x^i\right\rangle\right| \\
&> C\sum_{j=0}^{N}\binom{N}{j}\cdot\max_{l\le N}\sup_{|x|\le 1}\left|\theta_0^{(l)}(x)\right|\cdot\left|\sum_{i=N+1}^{N_1}\frac{\phi^{(i)}(0)}{i!}\right|\cdot 2^{-M} \\
&\ge C\sum_{j=0}^{N}\binom{N}{j}\cdot\max_{l\le N}\sup_{|x|\le 2^{-M}}\left|\theta_M^{(l)}(x)\right|\cdot\left|\sum_{i=N+1}^{N_1}\frac{\phi^{(i)}(0)}{i!}\right|\cdot 2^{-M(N+1)} \\
&\ge C\sum_{j=0}^{N}\binom{N}{j}\cdot\max_{l\le N}\sup_{|x|\le 2^{-M}}\left|\theta_M^{(l)}(x)\right|\cdot\max_{l\le N}\sup_{|x|\le 2^{-M}}\left|\sum_{i=N+1}^{N_1}\frac{\phi^{(i)}(0)}{i!}x^i\right| \\
&\ge C\max_{l\le N}\sup_{|x|\le 2^{-M}}\left|\sum_{j=0}^{l}\binom{l}{j}\theta_M^{(l-j)}(x)\left(\sum_{i=N+1}^{N_1}\frac{\phi^{(i)}(0)}{i!}x^i\right)^{(j)}\right| \\
&= C\max_{l\le N}\sup_{|x|\le 2^{-M}}\left|\left(\theta_M(x)\sum_{i=N+1}^{N_1}\frac{\phi^{(i)}(0)}{i!}x^i\right)^{(l)}\right|.
\end{aligned}$$

This is a contradiction, and hence we obtain the conclusion. □

We here have a question whether Theorem 5 for distributions can be also proved. We will see in Theorem 10 that Theorem 5 for distributions is equivalent to BD-$\mathbb{N}$.

We in the rest of this section show an application of Theorem 4. A function $f : (a, b) \to \mathbb{R}$ is said to *vanish at end points* if for each k there exists m such that for all $x \in (a, b)$,

$$x < a + 2^{-m} \vee b - 2^{-m} < x \implies |f(x)| < 2^{-k}.$$

Note that if $f : (a, b) \to \mathbb{R}$ is a function which vanishes at end points and is uniformly continuous on each compact subinterval of (a, b), then there exists a uniformly continuous function $\widehat{f} : \mathbb{R} \to \mathbb{R}$ such that $\widehat{f} = f$ on (a, b) and $\widehat{f} = 0$ on $(-\infty, a) \cup (b, \infty)$; see [8, Proposition 1]. And note that, given functions $f, f' : (a, b) \to \mathbb{R}$ which vanish at end points, if f is uniformly differentiable on each compact subinterval of (a, b) with a derivative f', then $\widehat{f}$ is uniformly differentiable on $\mathbb{R}$ with a derivative $\widehat{f'}$; see [8, Proposition 2].

Proposition 6. *Let u be a distribution such that $\langle u, \phi\rangle \neq 0$ for some ϕ in $\mathcal{D}(\mathbb{R})$, and fix n in $\mathbb{N}$. Suppose that $\langle u, x^{n+1}\phi\rangle = 0$ for all ϕ in $\mathcal{D}(\mathbb{R})$. Then $\langle u, \phi\rangle = \sum_{i=0}^{n} \frac{\langle u, x^i\rangle}{i!}\phi^{(i)}(0)$ for all ϕ in $\mathcal{D}(\mathbb{R})$.*

Proof. We first show that $\operatorname{supp} u = \{0\}$. Since $|\langle u, \phi\rangle| > 0$ for some ϕ in $\mathcal{D}(\mathbb{R})$, we have $\operatorname{supp} u \cap \operatorname{supp} \phi \neq \emptyset$. Therefore we can take s in $\operatorname{supp} u$ by [12, Cororollary 5.2]; see the last paragraph of Section 2. Assume that $|s| > 0$. Then we can choose ϕ in $\mathcal{D}(\mathbb{R})$ such that $\operatorname{supp} \phi \subset (s - |s|/2, s + |s|/2)$, by the definition of $\operatorname{supp} u$. Define a function $\psi : (-\infty, s - |s|/2] \cup (s - |s|/2, s + |s|/2) \cup [s + |s|/2, \infty) \to \mathbb{R}$ as follows: set $\psi(x) := \frac{1}{x^{n+1}}\phi(x)$ if $|x - s| < \frac{|s|}{2}$; and set $\psi(x) := 0$ if $|x - s| \geq \frac{|s|}{2}$. Then ψ is uniformly differentiable function on $(s - |s|/2, s + |s|/2)$, and vanishes at the end points $s - |s|/2$ and $s + |s|/2$. We therefore obtain a test function $\widehat{\psi}$ which is an extension of ψ to $\mathbb{R}$ from [8, Propositions 1 and 2]; see the foregoing notes. It is easy to show that $x^{n+1}\widehat{\psi}$ is also a test function and that $x^{n+1}\widehat{\psi} = \phi$. We then have $0 < |\langle u, \phi\rangle| = \left|\left\langle u, x^{n+1}\widehat{\psi}\right\rangle\right| = 0$, a contradiction. Thus $s = 0$.

We now let k be in $\mathbb{N}$, and ϕ in $\mathcal{E}(\mathbb{R})$. Then there exists $M \geq n$ such that $\left|\langle u, \phi\rangle - \sum_{i=0}^{m} \frac{\langle u, x^i\rangle}{i!}\phi^{(i)}(0)\right| < 2^{-k}$ for all $m \geq M$, by Theorem 4. Note that for any $i \geq n + 1$, $\langle u, x^i\rangle = \langle u, \theta_1 x^i\rangle = \langle u, x^{n+1} \cdot \theta_1 x^{i-(n+1)}\rangle = 0$, by Lemma 2 and the supposition. That is, $\frac{\langle u, x^i\rangle}{i!} = 0$ for all $i \geq n + 1$. Thus we have $\sum_{i=0}^{m} \frac{\langle u, x^i\rangle}{i!}\phi^{(i)}(0) = \sum_{i=0}^{n} \frac{\langle u, x^i\rangle}{i!}\phi^{(i)}(0)$, and hence $\left|\langle u, \phi\rangle - \sum_{i=0}^{n} \frac{\langle u, x^i\rangle}{i!}\phi^{(i)}(0)\right| < 2^{-k}$. Since k is arbitrary, we conclude $\langle u, \phi\rangle = \sum_{i=0}^{n} \frac{\langle u, x^i\rangle}{i!}\phi^{(i)}(0)$. □

4 Continuity of distributions and BD-$\mathbb{N}$

We first show that every mapping of $\mathbb{N}$ into $\mathbb{N}$ preserves pseudoboundedness.

A mapping α of $\mathbb{N}$ into $\mathbb{N}$. is *strictly increasing* if $\alpha(n + 1) > \alpha(n)$ for all n. It is clear that a strictly increasing mapping α satisfies that $\alpha(n) \geq n$ for all n.

Lemma 7. *Let A be a subset of $\mathbb{N}$, and α a strictly increasing mapping of $\mathbb{N}$ into $\mathbb{N}$. Then A is pseudobounded if and only if for any $\{a_n\}$ is a sequence in A, there exists N in $\mathbb{N}$ such that $a_n < \alpha(n)$ for all $n \geq N$.*

Proof. We show the part "only if". Let $\{a_n\}$ be a sequence in A. Then there exists N such that $a_n < n$ for all $n \geq N$. Since $\alpha(n) \geq n$ for all n, we have that $a_n < n \leq \alpha(n)$ for all $n \geq N$.

We next show the part "if". Let $\{a_n\}$ be any sequence in A. Construct a binary sequence $\{\lambda_n\}$ such that

$$\lambda_n = 0 \Longrightarrow \max\{\frac{a_k}{k} | \alpha(n) \leq k < \alpha(n+1)\} < 1,$$
$$\lambda_n = 1 \Longrightarrow \max\{\frac{a_k}{k} | \alpha(n) \leq k < \alpha(n+1)\} \geq 1.$$

Define a sequence $\{b_n\}$ as follows. If $\lambda_n = 0$, then set $b_n := a_0$. If $\lambda_n = 1$, then there exists k such that $\alpha(n) \leq k < \alpha(n+1)$ and $\frac{a_k}{k} \geq 1$, and set $b_n := a_k$. Then $\{b_n\}$ is a sequence in A, and therefore there exists N such that $b_n < \alpha(n)$ for all $n \geq N$, by the assumption. If there exists $n \geq N$ such that $\lambda_n = 1$, then there exists k such that $\alpha(n) \leq k \leq a_k \equiv \alpha(n)$, a contradiction. Thus $\lambda_n = 0$ for all $n \geq N$, and hence $a_n < n$ for all $n \geq \alpha(N)$. □

Lemma 8. *Let A be a pseudobounded subset of $\mathbb{N}$, and B a subset of $\mathbb{N}$. If, for each n in B, there exists m in A with $n \leq m$, then B is pseudobounded.*

Proof. Let $\{b_n\}$ be any sequence in B. Construct a sequence $\{a_n\}$ in A such that $b_n \leq a_n$ for each n. Then there exists N such that $\frac{b_n}{n} \leq \frac{a_n}{n} < 1$ for all $n \geq N$. □

Lemma 9. *Let A be a pseudobounded subset of $\mathbb{N}$, and α be a mapping of $\mathbb{N}$ into $\mathbb{N}$. Then the image $\alpha(A)$ is pseudobounded.*

Proof. We first assume that α is strictly increasing. Let $\{\alpha(a_n)\}$ be any sequence in $\alpha(A)$. Choose N in $\mathbb{N}$ such that $a_n < n$ for all $n \geq N$. Then, since α is strictly increasing, $\alpha(a_n) < \alpha(n)$ for all $n \geq N$. Thus $\alpha(A)$ is pseudobounded by Lemma 7.

We consider the general case. Let $\{\alpha(a_n)\}$ be any sequence in $\alpha(A)$. Set $\beta(n) := \max_{i \leq n} \{\alpha(i), n\}$ for each n. Then β is strictly increasing, and therefore $\beta(A)$ is pseudobounded by the former argument. Also, β satisfies that $\alpha(n) \leq \beta(n)$ for all n. Thus $\alpha(A)$ is pseudobounded by Lemma 8. □

We now give the main theorem in this paper.

Theorem 10. *The following are equivalent.*

1. *Every distribution is bounded.*
2. *The Banach-Steinhaus theorem for bounded distributions: for a sequence $\{u_n\}$ of bounded distributions, if $\langle u, \phi\rangle \equiv \lim_n \langle u_n, \phi\rangle$ exists for all ϕ in $\mathcal{D}(\mathbb{R})$, then u is a bounded distributions.*
3. *Let u be a distribution whose support consists of the origin. Then there exists N in $\mathbb{N}$ such that $\langle u, \phi\rangle = \sum_{i=0}^{N} \frac{\langle u, x^i\rangle}{i!} \phi^{(i)}(0)$ for all ϕ in $\mathcal{D}(\mathbb{R})$.*
4. *Let u be a distribution whose support consists of the origin. Then for each k, there exists N in $\mathbb{N}$ such that*
$$\left| \langle u, \phi\rangle - \sum_{i=0}^{n} \frac{\langle u, x^i\rangle}{i!} \phi^{(i)}(0) \right| < 2^{-k} \quad (n \geq N, \phi \in \mathcal{D}(\mathbb{R})).$$
5. BD-$\mathbb{N}$.

Proof. (5) $\Longrightarrow$ (1). Every $\mathcal{D}_k(\mathbb{R})$ is a separable metric space (see [12, Proposition 4.5]). We have the facts that a linear functional on $\mathcal{D}(\mathbb{R})$ is sequentially continuous if and only if it is on each $\mathcal{D}_k(\mathbb{R})$, and that a linear functional on $\mathcal{D}(\mathbb{R})$ is bounded if and only if it is on each $\mathcal{D}_k(\mathbb{R})$, from [12, Proposition 4.2 and Corollary 4.3]. This implication thus follows from [5, Theorem 4] (see Section 1).

(1) $\Longrightarrow$ (2) follows from the Banach-Steinhaus theorem for distributions; see [10, Theorem 4.10] and Section 1.

(2) $\Longrightarrow$ (3). Let u be a distribution whose support consists of the origin. We then have the series $\left\{\sum_{i=0}^{n} \frac{\langle u, x^i\rangle}{i!}\phi^{(i)}(0)\right\}_n$ of bounded distributions converging to $\langle u, \phi\rangle$, from Theorem 4, and therefore u is bounded by the hypothesis. Using Theorem 5, we obtain the conclusion.

(3) $\Longrightarrow$ (4). Let u be a distribution whose support consists of the origin. By the hypothesis, we have $\langle u, \phi\rangle = \sum_{i=0}^{N} \frac{\langle u, x^i\rangle}{i!}\phi^{(i)}(0)$ for some N, and therefore, for all ϕ in $\mathcal{D}(\mathbb{R})$,

$$\begin{aligned} |\langle u, \phi\rangle| &\leq \sum_{i=0}^{N} \left|\frac{\langle u, x^i\rangle}{i!}\right| \cdot \left|\phi^{(i)}(0)\right| \leq \sum_{i=0}^{N} \left|\frac{\langle u, x^i\rangle}{i!}\right| \max_{l \leq N} \sup_{|x| \leq 1} \left|\phi^{(l)}(x)\right| \\ &\leq \sum_{i=0}^{N} \left|\frac{\langle u, x^i\rangle}{i!}\right| \max_{l \leq n} \sup_{|x| \leq 1} \left|\phi^{(l)}(x)\right| \quad (n \geq N). \end{aligned}$$

We then have $\langle u, \phi\rangle = \sum_{i=0}^{n} \frac{\langle u, x^i\rangle}{i!}\phi^{(i)}(0)$ for all $n \geq N$, by Theorem 5.

(4) $\Longrightarrow$ (5). Let A be a countable pseudobounded subset of $\mathbb{N}$, and $\{a_k\}$ an enumeration of all elements of A. We may assume that $a_0 = 0$ and that $\{a_k\}$ is increasing. Given any test function ϕ, construct a sequence $\{s_k\}$ in $\mathbb{N}$ as follows. Set $s_0 := a_0 = 0$. For each $k \geq 1$, there exist two ms in $\mathbb{N}$ such that $m - 1 < \left|\phi^{(a_k)}(0)\right| < m + 1$; set $s_k := \max\left\{m \in \mathbb{N} : \left|\phi^{(a_k)}(0) - m\right| < 1\right\}$. Then $\{s_k\}$ is a pseudobounded set by Lemma 9. It follows from this matter that we can choose a strictly increasing sequence $\{K_n\}$ such that for all n,

$$\left|\frac{\phi^{(a_k)}(0)}{2^k}\right| \leq \frac{s_k}{k} < 2^{-n} \quad (k \geq K_n).$$

We have

$$\sum_{i=k+1}^{m} \left|\frac{\phi^{(a_{K_i})}(0)}{2^{K_i}}\right| < \sum_{i=k+1}^{m} 2^{-i} < 2^{-k} \quad (m > k).$$

Set

$$\langle u_n, \phi\rangle := \sum_{i=0}^{n} \frac{\phi^{(a_{K_i})}(0)}{2^{K_i}} \quad (n \in \mathbb{N}, \phi \in \mathcal{D}(\mathbb{R})).$$

Then, given any ϕ, the sequence $\{\langle u_n, \phi\rangle\}_n$ is a Cauchy sequence in $\mathbb{R}$, and therefore converges in $\mathbb{R}$. Note that $\operatorname{supp} u_n = \{0\}$ for all n. We can here set

$$\langle u, \phi\rangle := \sum_{i=1}^{\infty} \frac{\phi^{(a_{K_i})}(0)}{2^{K_i}} \quad (\phi \in \mathcal{D}(\mathbb{R})).$$

Then u is a distribution, by the Banach-Steinhaus theorem for distributions; see [10, Theorem 4.10]. It is clear that $\operatorname{supp} u = \{0\}$, and therefore u is extended to $\mathcal{E}(\mathbb{R})$.

Now choose N such that for all $n \geq N$,

$$\left| \langle u, \phi \rangle - \sum_{i=0}^{n} \frac{\langle u, x^i \rangle}{i!} \phi^{(i)}(0) \right| < 1 \quad (\phi \in \mathcal{E}(\mathbb{R})),$$

by the hypothesis. If there exists n with $a_{K_n} > N$, then, letting $\phi(x) := \frac{2^{K_n}}{a_{K_n}!} x^{a_{K_n}}$, we have

$$\begin{aligned} 1 &> \left| \langle u, \phi \rangle - \sum_{i=0}^{N} \frac{\langle u, x^i \rangle}{i!} \phi^{(i)}(0) \right| = \left| \sum_{i=1}^{\infty} \frac{\phi^{(a_{K_i})}(0)}{2^{K_i}} - \sum_{i=0}^{N} \frac{\phi^{(i)}(0)}{i!} \right| = \sum_{i=1}^{\infty} \frac{\phi^{(a_{K_i})}(0)}{2^{K_i}} \\ &\geq \frac{\phi^{(a_{K_n})}(0)}{2^{K_n}} = 1, \end{aligned}$$

a contradiction. Thus $a_{K_n} \leq N$ for all n, and hence $a_k \leq N$ for all k, since $\{K_n\}$ is strictly increasing and since $\{a_k\}$ is increasing. □

5 On $(\mathcal{F})$-spaces and $(\mathcal{LF})$-spaces

We in the rest of this paper give notes on $(\mathcal{F})$-spaces and $(\mathcal{LF})$-spaces, with respect to Theorem 10.

Theorem 11. *The following are equivalent.*

1. *Every sequentially continuous linear functional on a separable locally convex space with countable seminorms is bounded.*
2. *Every sequentially continuous linear functional on a separable $(\mathcal{F})$-space is bounded.*
3. *The Banach-Steinhaus theorem for bounded linear functionals on a separable $(\mathcal{F})$-space: for a sequence $\{u_n\}$ of bounded linear functionals on a separable $(\mathcal{F})$-space, if $u(x) \equiv \lim_n u_n(x)$ exists for each x in X, then the limit u is bounded.*
4. BD-$\mathbb{N}$.

Proof. (4) $\implies$ (1). A locally convex space with countable seminorms is a metric space; see [11, Proposition 1.1]. A pointwise continuous linear functional on a locally convex space is bounded. BD-$\mathbb{N}$ implies that every sequentially continuous mapping of a separable metric space into a metric space is pointwise continuous, as found in [5, Proposition 2]. We therefore obtain the implications.

(1) $\implies$ (2) is trivial. (2) $\implies$ (3) follows from the Banach-Steinhaus theorem for sequentially continuous linear functionals on a $(\mathcal{F})$-space, given in [12, Proposition 2.4].

(3) $\implies$ (4). The hypothesis implies the Banach-Steinhaus theorem for bounded linear functionals on the space $\mathcal{E}(\mathbb{R})$, which is equivalent to BD-$\mathbb{N}$; see [12, Theorem 7.3]. □

Theorem 12. *The following are equivalent.*

1. *Every sequentially continuous linear functional on a $(\mathcal{LF})$-space with a sequence of separable $(\mathcal{F})$-spaces is bounded.*
2. *The Banach-Steinhaus theorem for bounded linear functionals on a $(\mathcal{LF})$-space with a sequence of separable $(\mathcal{F})$-spaces.*

3. BD-ℕ.

Proof. Let X be a $(\mathcal{LF})$-space with the sequence $\{X_k\}$ of separable $(\mathcal{F})$-spaces.

(3) $\implies$ (1). Let u be any sequentially continuous linear functional on X. Then u is a sequentially continuous linear functional on each X_k, since for each k, if a sequence $\{x_n\}$ converges to 0 in X_k, then it converges to 0 in X. It follows from the hypothesis and Theorem 11 that u is bounded on each X_k. Thus u is bounded on X from [12, Prposition 4.2].

(3) $\implies$ (2). Let $\{u_n\}$ be a sequence of bounded linear functionals on X with the sequence $\{X_k\}$. Then, on each X_k, every u_n is bounded, and therefore is sequentially continuous. Assume that $u(x) \equiv \lim_n u_n(x)$ exists for each x in X. Then, u is sequentially continuous on each X_k, by the Banach-Steinhaus theorem for $(\mathcal{F})$-spaces (see [12, Theorem 2.3]). Therefore the hypothesis implies that u is bounded on each X_k, by Theorem 11. Thus u is bounded on X.

(1) $\implies$ (3) and (2) $\implies$ (3) follow from Theorem 10. □

Given any $(\mathcal{LF})$-space X with a sequence $\{X_k\}$ of separable $(\mathcal{F})$-spaces, we have not known whether or not the following two propositions are provable yet: every sequentially continuous linear functional on each X_k is sequentially continuous on X; the Banach-Steinhaus theorem for sequentially continuous linear functionals on X.

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