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Powers of positive elements in constructive C^*-algebras

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Hiroki TAKAMURA

Abstract

In this note we show that Ogasawa's theorem has a constructive proof in **BISH** of Bishop's constructive mathematics which means we use intuitionistic logic rather than classical logic, together with some suitable foundation. Ogasawara's theorem is about powers of positive elements in C^*-algenras. In [17], the author introduces an elementary theory of C^*-algebras in **BISH**, however powers of positive elements in constructive C^*-algebras is not deeply treated. In this note we investigate that constructive treatment of Ogasawa's theorem.

1 Introduction

In [13], Ogasawara has shown that the following theorems that are about powers of positive elements in C^*-algebras. Let $\mathfrak{A}$ be a C^*-algebra. Then for $x, y \in \mathfrak{A}_+$

- if $0 \leq x \leq y$ then $0 \leq x^{1/2} \leq y^{1/2}$
- if $0 \leq x \leq y$ implies always $0 \leq x^2 \leq y^2$ then $\mathfrak{A}$ is commutative.

In [14], these statements are generalised, however these proofs proceeds based on the term of the spectrum of an element. The spectrum of an element is difficult to treat in Bishop style constructive mathmatics, indeed [7], there are Brouwerian examples that we cannot establish elementary properties of spectra which are obtained in classical mathematics. Hence, these proofs are not acceptable in **BISH**. In this note, we will show that these statement has constructive proofs in **BISH**.

Related works are as follows. Bishop [2] described commutative Banach algebras and a special case of commutative C^*-algebras which consists of normable operators on a Hilbert space in constructive mathematics. In his book, he proves a constructive version of the Gelfand representation theorem for the special case mentioned above. These results appear in Bishop and Bridges [3], and Bridges [5]. In Bridges [6] and Havea's PhD thesis [9], the authors study constructive non-commutative Banach algebras. In Spitters' PhD thesis [16], he claims that the proof of the Gelfand representation theorem in Bishop [2] also works well in the general setting of a commutative C^*-algebra and he gives a constructive proof of the theorem. The area of constructive von Neumann algebras is studied by Vîţă in her PhD thesis [19], and also by Spitters [16].

In this note, we assume familiarity with constructive mathematics, as found in [2, 3, 5], and with the theory of C^*-algebra, as found in [4, 12, 14, 15, 18, 20], also with the constructive treatment for operator algebras, as founde in [9, 19, 16].

2 Constructive C^*-algebras

An *involution* on an algebra $\mathfrak{A}$ over a field $\mathbb{K}$ is a map $^* : \mathfrak{A} \to \mathfrak{A}$ such that for all $x, y \in \mathfrak{A}$ and $a, b \in \mathbb{K}$,

$$(xy)^* = y^* x^*,$$

$$(x^*)^* = x,$$

$$(ax + by)^* = \overline{a} x^* + \overline{b} y^*,$$

where $\overline{a}, \overline{b}$ are conjugates of a and b.

A $*$-*algebra* over a field $\mathbb{K}$ is an algebra with an involution. Let $\mathfrak{A}$ be an algebra over a complex field $\mathbb{C}$ with unit $\mathbf{e}$. An algebra $\mathfrak{A}$ is called a *Banach algebra* if $\mathfrak{A}$ is a Banach space with $||\mathbf{e}|| = 1$, and $||xy|| \leq ||x|| ||y||$ for all x, y in $\mathfrak{A}$.

Note that by definition a Banach space in constructive mathematics is separable, hence a Banach algebra is separable.

Definition 1 (Constructive C^*-algebra)**.** A Banach $*$-algebra $\mathfrak{A}$ is called a C^*-*algebra* if $||x^* x|| = ||x||^2$ for all x in $\mathfrak{A}$.

If $\mathcal{H}$ is a Hilbert space, we denote by $\mathfrak{B}(\mathcal{H})$ the set of bounded linear operators on $\mathcal{H}$. An operator $T \in \mathfrak{B}(\mathcal{H})$ is said to be *compact* if $\{T(x) : ||x|| \leq 1\}$ is totally bounded. Every compact operator is normable and has its adjoint. The set of all compact operators on a Hilbert space is an example of constructive C^*-algebra.

Definition 2 (Concrete C^*-algebra)**.** A self-adjoint $*$-subalgebra $\mathfrak{R}$ of normable elements of $\mathfrak{B}(\mathcal{H})$ is called *concrete C^*-algebra* if it is complete and separable with respect to the norm.

Classically, the set $\mathfrak{B}(\mathcal{H})$ of all bounded linear operators on a Hilbert space $\mathcal{H}$ is an example of C^*-algebra. However, $\mathfrak{B}(\mathcal{H})$ is not an example of constructive C^*-algebra, since, as the following Brouwerian example [11] shows that every bounded linear operator in $\mathfrak{B}(\mathcal{H})$ is not normable.

Let $\{a_n\}$ be a binary sequence and $\{e_n\}$ a orthonormal basis for ℓ^2. Define a linear operator A on ℓ^2 by

$$Ax = \sum_{n=1}^{\infty} a_n x_n e_n,$$

where x_n is the nth component of x. Then it is easy to see that A is a self-adjoint. If A is normable, then either $||A|| > 0$ in which case $a_n = 1$ for some n, or else $||A|| < 1$ and $a_n = 0$ for all n. Thus, the statement

Every bounded operator on a Hilbert space has a norm

entails the Limit Principle of Omniscience (LPO).

3 Positive elements

In this section, we study positive elements in constructive C^*-algebra. An element x in a C^*-algebra $\mathfrak{A}$ is *normal* if $x^*x = xx^*$ and is *self-adjoint* if $x = x^*$ holds. Trivially, any self-adjoint element is a normal element: for any x in $\mathfrak{A}$ there exist self-adjoint elements x_1, x_2 such that $x = x_1 + \imath x_2$, where $x_1 = (x + x^*)/2, x_2 = (x^* - x)/2\imath$. Let $\mathfrak{A}$ be a C^*-algebra. For each $x \in \mathfrak{A}$, the set

$$Sp(x) = \{\lambda \in \mathbb{C} : (x - \lambda \mathbf{e})^{-1} \text{ does not exist } \},$$

is called the *spectrum of* x.

The following classical result can be found on page 6 of [14].

Theorem The following conditions on an element in a C^*-algebra $\mathfrak{A}$ are equivalent.

(1) x is normal and $Sp(x) \subset [0, \infty)$.
(2) There is a self-adjoint element y in $\mathfrak{A}$ such that $x = y^2$.
(3) There is an element y in $\mathfrak{A}$ such that $x = y^*y$.
(4) x is self-adjoint and $||a\mathbf{e} - x|| \leq a$ for any $a \geq ||x||$.
(5) x is self-adjoint and $||a\mathbf{e} - x|| \leq a$ for some $a \geq ||x||$.

Classically, an element x in a C^*-algebra $\mathfrak{A}$ is *positive* if it satisfies any of the five conditions (1) $\sim$ (5).

In the classical theory, condition (1) plays an essential role. However, the spectrum of an element is not easy to treat constructively. In [7], there are Brouwerian examples that we cannot establish elementary properties of spectra which are obtained in classical mathematics. Hence, we need to modify the condition (1) of a positive element to show the equivalence in constructive C^*-algebra.

The spectrum Σ of a commutative Banach algebra $\mathfrak{A}$ consists of all nonzero bounded multiplicative linear functionals. Each element in the spectrum Σ of $\mathfrak{A}$ is called a *character*. With regard to commutative constructive C^*-algebras, the Gelfand representation theorem states that every commutative C^*-algebra is isomorphic to the space of continuous functions on its spectrum.

Note that the spectrum of a Banach algebra is compact classically. Constructively, this is not true for Banach algebras [6, 9], but it is true for C^*-algebras.

Theorem 1 (Gelfand representation theorem [16])**.** *Let $\mathfrak{A}$ be a commutative C^*-algebra and let $\mathsf{C}(\Sigma)$ be the set of all complex valued continuous functions on the spectrum Σ of $\mathfrak{A}$. Then there exists a norm preserving $*$-isomorphism from $\mathfrak{A}$ onto $\mathsf{C}(\Sigma)$.*

Let $\mathfrak{A}$ be a C^*-algebra and x a normal element in $\mathfrak{A}$. Then $[\mathbf{e}, x]$ denotes the commutative C^*-subalgebra of $\mathfrak{A}$ generated by $\mathbf{e}$ and x and $\mathsf{C}(\Sigma)$ denotes the set of all complex valued continuous functions on the spectrum Σ of $[\mathbf{e}, x]$. For a norm preserving $*$-isomorphism ϕ from $[\mathbf{e}, x]$ onto $\mathsf{C}(\Sigma)$ (the existence of such ϕ is assured by Theorem 1), we say that $\phi(x)$ is a *name* of x. A name $\phi(x)$ of x is nonnegative if it satisfies $\phi(x)u \geq 0$ for all $u \in \Sigma$.

In the rest of the paper, unless otherwise specified we use the symbol ϕ as a particular $*$-isomorphism from $[\mathbf{e}, x]$ onto $\mathsf{C}(\Sigma)$.

Definition 3 (Positive element)**.** An element x in a C^*-algebra $\mathfrak{A}$ is called *positive* if $\phi(x)$ is nonnegative.

Since $[\mathbf{e}, x]$ is defined for a normal element x, and the name $\phi(x)$ of x is nonnegative, it is easy to see that a positive element is self-adjoint.

$\mathfrak{A}_{sa}$ and $\mathfrak{A}_+$ denote the set of all self-adjoint elements in a C^*-algebra $\mathfrak{A}$ and the set of all positive elements in $\mathfrak{A}$ respectively. We also denote $-\mathfrak{A}_+ = \{x \in \mathfrak{A} : -x \in \mathfrak{A}_+\}$ and $\mathfrak{A}_{+1} = \{x \in \mathfrak{A}_+ : ||x|| \leq 1\}$. Let x, y in $\mathfrak{A}_{sa}$, then we write $x \geq y$ for $x - y$ is in $\mathfrak{A}_+$.

Lemma 2. *Let $\mathfrak{A}$ be a C^*-algebra and x in $\mathfrak{A}$. The following conditions are equivalent constructively.*

(1) *x is a positive element in $\mathfrak{A}$.*

(2) *x is a self-adjoint element and $||a\mathbf{e} - x|| \leq a$ for some $||x|| \leq a$.*

By Lemma 2, we can show that if x, y are positive elements in $\mathfrak{A}$ then $x + y$ is also positive. Indeed, $||(||x|| + ||y||)\mathbf{e} - (x + y)|| \leq ||x|| + ||y||$. So we take $a = ||x|| + ||y|| \geq ||x + y||$.

Note that the following condition (i) is also equivalent to the definition of positivity.

(i) $x = x^*$ and $|||x||\mathbf{e} - x|| \leq ||x||$.

Moreover, the condition (i) is equivalent to the following condition.

(ii) $x = x^*$ and $||a\mathbf{e} - x|| \leq a$ for all $||x|| \leq a$.

Indeed, it is clear that (ii) implies (i). Conversely, if x satisfies (i), then $||a\mathbf{e} - x|| = ||(a - ||x||)\mathbf{e} + ||x||\mathbf{e} - x|| \leq a - ||x|| + |||x||\mathbf{e} - x|| \leq a - ||x|| + ||x|| = a$. So we have (ii).

The following comes from (1.4.1) on page 7 of [15]. We give a proof without using the spectrum of an element.

Lemma 3. *Let x be a positive element in a C^*-algebra $\mathfrak{A}$. Then for any positive integer n, there exists an unique positive element y in $\mathfrak{A}$ such that $y^n = x$. This y is denoted by $x^{1/n}$.*

Proof. Since $\phi(x)$ is nonnegative in $\mathsf{C}(\Sigma)$, we can define $g(x)$ by $\phi(x)^{1/n}$. Then $g(x)$ is also nonnegative and $g(x) = g(x^*)$. Thus $\phi^{-1}(g(x))$ is a positive element of $\mathfrak{A}$. Now we denote $\phi^{-1}(g(x))$ by y. To see the uniqueness, assume that $y'^n = x$ for some other $y' \in \mathfrak{A}_+$. Since $y'x = xy'$ and $[\mathbf{e}, x]$ is isomorphic to $\mathsf{C}(\Sigma)$, y can be expressed by the limit of some polynomial of x. Then we have $yy' = y'y$. Let ϕ' be a norm preserving $*$-isomorphism from $[\mathbf{e}, y, y']$ onto $\mathsf{C}(\Sigma')$. By assumption, $\phi'(y)^n = \phi'(y')^n = \phi'(x)$ and $\phi'(y), \phi'(y')$ is nonnegative. Then $(\phi'(y)^n - \phi'(y')^n)u = 0$ for any $u \in \Sigma'$. Thus $(\phi'(y) - \phi'(y'))u = 0$. Hence $\phi'(y) = \phi'(y')$. Since ϕ' is a norm preserving $*$-isomorphism, we conclude that $y = y'$.

The proof of the following results holds constructively and can be found in [12].

Lemma 4. *Let x be a self-adjoint element in a C^*-algebra $\mathfrak{A}$. Then there exist unique positive elements x_+ and x_- such that $x = x_+ - x_-$, $x_+x_- = 0$ and $||x|| = \max\{||x_+||, ||x_-||\}$.*

We say that x_+ and x_- are *decompositions* of x.

We are now in position to give the constructive analogue of Theorem and the proofs can be found of [17].

Theorem 5. *The following conditions are equivalent constructively.*

(1) x is a positive element in a C^*-algebra $\mathfrak{A}$.
(2) There is a self-adjoint element y in $\mathfrak{A}$ such that $x = y^2$.
(3) There exists an element y in $\mathfrak{A}$ such that $x = y^*y$.
(4) x is self-adjoint and $||a\mathbf{e} - x|| \leq a$ for any $a \geq ||x||$.
(5) x is self-adjoint and $||a\mathbf{e} - x|| \leq a$ for some $a \geq ||x||$.

The following are some properties of elements in C^*-algebra. First, it is easy to show the equivalence $x = 0$ and $x^*x = 0$ (exercise 8.6 on page 54 of [20])

The classical counterpart of the next proposition holds constructively. For the sake of completeness, we will give the proof of the proposition.

Proposition 6. *For any elements x, y, z in a C^*-algebra $\mathfrak{A}$, the following properties hold:*

(1) *If $x \geq y \geq 0$ then $||x|| \geq ||y||$.*
(2) *If $x \geq 0$ then $||x||x \geq x^2$.*
(3) *If $x \geq y$ then $z^*xz \geq z^*yz$.*
(4) *If $x \geq y \geq 0$ then $x^{-1} \leq y^{-1}$.*

Proof. (1) It is easy to see that if x is positive then $||x||\mathbf{e}-x$ is also positive. Suppose $x \geq y \geq 0$, then $||x||\mathbf{e}-y$ is positive. Then, $\phi(||x||\mathbf{e}-y) = ||x||-\phi(y)$ is nonnegative. On the other hand $||y|| = \sup\{|\phi(y)u| : u \in \Sigma\}$. Hence $||x|| \geq ||y||$.

(2) It is easy to see that $\phi(||x||x-x^2)$ is positive and so $||x||x-x^2$ is positive.

(3) Suppose $x \geq y$, then $x-y$ is positive. By Theorem 7, we can write $x-y = w^*w$ for some w in $\mathfrak{A}$. Then, $z^*xz - z^*yz = z^*(w^*w)z = (wz)^*wz$. Thus, $z^*xz \geq z^*yz$.

(4) Suppose $x \geq y$, then $x^{-1/2}xx^{-1/2} \geq x^{-1/2}yx^{-1/2}$ by (3). Thus, $\mathbf{e} \geq x^{-1/2}yx^{-1/2}$ and so $1 \geq ||y^{1/2}x^{-1/2}||$ by (1).

$||y^{1/2}x^{-1}y^{-1/2}|| = ||y^{1/2}x^{-1/2}||||(y^{1/2}x^{-1/2})^*|| \leq 1$. Therefore $y^{1/2}x^{-1}y^{1/2} \leq \mathbf{e}$ and so $x^{-1} \leq y^{-1}$.

4 Powers of positive elements in constructive C*-algebras

In [13], Ogasawara has shown that the following theorems that are about powers of positive elements in C^*-algebras. Let $\mathfrak{A}$ be a C^*-algebra. Then for $x, y \in \mathfrak{A}_+$

- if $0 \leq x \leq y$ then $0 \leq x^{1/2} \leq y^{1/2}$
- if $0 \leq x \leq y$ implies always $0 \leq x^2 \leq y^2$ then $\mathfrak{A}$ is commutative.

In [14], these statements are generalised as follows

- if $0 \leq x \leq y$ then $0 \leq x^\alpha \leq y^\alpha$ for any α with $0 < \alpha \leq 1$
- if $0 \leq x \leq y$ implies always $0 \leq x^\alpha \leq y^\alpha$, for some $\alpha > 1$ then $\mathfrak{A}$ is commutative.

However these proofs are not accepted in Bishop's constructive mathematics. In this section, we will show that these statement has constructive proofs in Bishop style with some modifications.

4.1 First case

First we will give a constructive proof of if $0 \leq x \leq y$ then $0 \leq x^\alpha \leq y^\alpha$ for any α with $0 < \alpha \leq 1$.

First we need modify the property operator monotonicity, which is classically defined the term of spectrum. However, the spectrum of an element is difficult to treat in constructive mathematics. So we define operator monotonicity without using the term of the spectrum of an element.

Definition 4 (operator monotone)**.** An operator f on an intervel in $[a,b] \subseteq \mathbb{R}$ is called operator monotone (increasing) if changing the domain $[a,b]$ of f to $\mathfrak{A}$, then f can also be considered an operator on $\mathfrak{A}$ with

$$x \leq y \Longrightarrow f(x) \leq f(y)$$

whenever $[-||y||, ||y||] \subseteq [a,b]$ in $\mathbb{R}$

Remark that the inverse-operator is operator monotone decreasing operator.

4.1.1 An example of operator monotone

For any $\alpha > 0$, we define the function f_α on $(-1/\alpha, \infty)$ by

$$f_\alpha(t) = (1 + \alpha t)^{-1} t.$$

It is easy to see that f_α is an example of an operator monotone increasing on $(-1/\alpha, \infty)$.

Proposition 7. *If $0 < \beta \leq 1$ the function $t \to t^\beta$ is operator monoton increasing on $\mathbb{R}^+$.*

Proof. If $0 \leq x \leq y$ then $f_\alpha(x) \leq f_\alpha(y)$ with f_α is defined above. Now consider the integration

$$\int_0^\infty f_\alpha(t) \alpha^{-\beta} d\alpha$$

$$= \int_0^\infty (1 + \alpha t)^{-1} t \alpha^{-\beta} d\alpha$$

Put $a = \alpha t$ we have

$$= \int_0^\infty (1 + a)^{-1} t a^{-\beta} t^\beta t^{-1} da$$

$$= t^\beta \int_0^\infty (1 + a)^{-1} a^{-\beta} da = \gamma t^\beta$$

with some $\gamma \in \mathbb{R}$. For all $t \in [0, ||y||]$ and ϵ there is a large n and an equidistant division

$$0 = \alpha_0 < \alpha_1 < \cdots < \alpha_m = n$$

of the intervel $[0, n]$ such that

$$|t^\beta - (\gamma m)^{-1} n \sum_{k=1}^{m} f_{\alpha_m}(t) \alpha_k^{-\beta}| < \epsilon.$$

It follows that $y^\beta - x^\beta \geq -2\epsilon$ and since ϵ is arbitrary, $x^\beta \leq y^\beta$

Theorem 8. *Let $\mathfrak{A}$ be a C^*-algebra. If $0 \leq x \leq y$ then $0 \leq x^\alpha \leq y^\alpha$ for any α with $0 < \alpha \leq 1$.*

4.2 Second case

Next, we will give a constructive proof of second case.

Theorem 9. *Let $\mathfrak{A}$ be a C^*-algebra. If $0 \leq x \leq y$ implies always $0 \leq x^\alpha \leq y^\alpha$, for some $\alpha > 1$ then $\mathfrak{A}$ is commutative.*

Proof. First remark that if α preserves order then so does α^n. Then using the previous theorem, wecan take arbitrary $r \in \mathbb{R}$ as the exponents. Therefore it suffices to prove the theorem with $\alpha = 2$.

Take $x, y \in \mathfrak{A}_+$ and $\epsilon > 0$. Then $x \leq x + \epsilon y$, thus $x^2 \leq (x + \epsilon y)^2$ by assumption. $0 \leq xy + yx + \epsilon y^2$ for any ϵ. we have

$$xy + yx \geq 0 \cdots \ (1)$$

Now put $xy = a + ib$, $a, b \in A_{sa}$. Then $2a = xy + yx \geq 0$. Thus, $a \in A_+$. However (1) is valid for any product of positive elements and

$$x(yxy) = a^2 - b^2 + i(ab + ba) \cdots \ (2),$$

$$y(xyx) = a^2 - b^2 - i(ab + ba)$$

From which we have $a^2 - b^2 \in A_+$.

Consider the set

$$E = \{\alpha \geq 1 | \alpha b^2 \geq a, x, y \in \mathfrak{A}_+, xy = a + ib\}$$

Note that $\alpha \in E$, then E is nonempty. Now we fix an element $\lambda \in E$. Then $a^2 - \lambda b^2 \geq 0$. Recall that $b \in \mathfrak{A}_{sa}$, so $b^2 \in \mathfrak{A}_+$. Using (1) we have

$$0 \leq b^2(a^2 - \lambda b^2) + (a^2 - \lambda b^2)b^2 = b^2a^2 + a^2b^2 - 2\lambda b^4 \cdots \ (3)$$

Recall that $a^2 - b^2 \geq 0$, $ab + ba \in \mathfrak{A}_{sa}$ then $(ab + ba)^2 \in \mathfrak{A}_+$. With (2) and the form of the element in the set E. Therefore we now have

$$\lambda(ab + ba)^2 \leq (a^2 - b^2)^2$$

Thus,

$$\lambda(ab^2a + ba^2b + a(bab) + (bab)a) \leq a^4 + b^4 - a^2b^2 - b^2a^2$$

On the left hand side we have $a(bab) + (bab)a \geq 0$ by (1) and $ba^2b \geq \lambda b^4$ by assumption and $ab^2a \geq 0$. Finally, we have the following inequation

$$a^4 + b^4 - 2\lambda b^4 \geq a^4 + b^4 - a^2b^2 - b^2a^2 \geq \lambda b^4$$

Therefore

$$a^4 + (1 - 2\lambda)b^4 \geq \lambda^2 b^4$$

Thus,

$$(\lambda^2 + 2\lambda - 1)b^4 \geq a^4$$

Using the previous theorem we have,

$$(\lambda^2 + 2\lambda - 1)^{1/2} b^2 \geq a^2.$$

This shows that If $\lambda \in E$ then $(\lambda^2 + 2\lambda - 1)^{1/2}$ is also in E.

Now we consider the sequence $\{a_i\}$ which is defined by

$$a_1 = \lambda, a_2 = (\lambda^2 + 2\lambda - 1)^{1/2}, \cdots, a_{n+1} = (a_n^2 + 2a_n - 1)^{1/2}$$

Since $a_n \geq (n+1)^{1/2}$, $\{a_i\}$ is a divergence sequence. Thenfore the set E is unbounded.

Thus, $\alpha b^2 \leq a$ for any α. It follows that $b = 0$. We conclude that $xy = yx$ for any $x, y \in \mathfrak{A}_+$.

Combine the facts that for any elements in $\mathfrak{A}$ can be expressed by self-adjoint elements and the Lemma 4, we finally obtain that $\mathfrak{A}$ is commutative.

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