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Algebraic Structure for a modal fixed point logic and abstract interpretation

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ALGEBRAIC STRUCTURE FOR A MODAL FIXED POINT LOGIC AND ABSTRACT INTERPRETATION

様相不動点論理と抽象解釈のための代数構造

by

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ABSTRACT

This thesis formulates a unified notion of abstractions as they appear in abstract interpretations and abstract models for model checking. Our aim is to clarify the relationship between the soundness and completeness theorems for interpretations of logics and their formula-preservation theorems. We mean by the latter that the truth of formulas under interpretations is preserved across abstractions. Our formulation is based on an algebraic approach.

For the formulation, we define a modal fixed point logic $R\mu$. The logic corresponds to a fragment of modal μ-calculus, which is expressive enough to contain the translation of all CTL-formulas.

The soundness and completeness for a logic is often proved from the fact that formulas form a free algebra for some algebraic structure, but finding one for a modal fixed point logic is in general difficult.

Our first contribution is to formulate the algebraic structure **RMu** that matches $R\mu$. To overcome the difficulty, we give **RMu** as an algebraic structure not on sets but on locally ordered categories (categories with partially ordered hom-sets). The signature Σ of a theory in $R\mu$ is seen as a locally ordered category. Interpretations of the theory are certain locally ordered functors from Σ to an **RMu**-algebra, which is a locally ordered category with enough structure to interpret $R\mu$. The free **RMu**-algebra construction gives soundness and completeness for this class of interpretations.

The second contribution is our 2-categorical framework in which the formula-preservation theorem becomes an extension of soundness and completeness theorems. We study the notion of abstraction relations as the 2-cells of a certain 2-category of locally ordered categories and functors, namely lax transformations whose components have left adjoints. We show in this setting that the formula-preservation theorem is derived from the same 2-categorical free **RMu**-algebra construction that gives soundness and completeness theorems.

Our formulation of abstraction uniformly justifies not only varieties of well-known abstractions, for example, simulations and bisimulations, but also a new abstraction from a Kripke structure to another structure.

We formulate a construction method of a complex interpretation of $R\mu$ from a simpler one in algebraic approach. The semantics of interpretations is based on the algebraic structure on functors for fibrations. The 2-categorical free algebra construction gives construction of not only interpretations but also abstractions.

This thesis formulates algebraic structures as generalised Lawvere theories and gives the unified construction of their free algebras. The above 2-categorical free algebra constructions are given as examples.

論文要旨

この論文では，抽象解釈やモデル検査の抽象モデルに現れる，抽象化という概念を定式化する．我々の定式化の目的は，論理の解釈の健全性や完全性と，論理式保存定理との関係を明らかにすることである．論理式保存定理とは，ある解釈の下での論理式の真偽が，抽象化を介して保存されるという定理である．我々は，代数的アプローチに基づいて解釈や抽象化を定式化する．

我々は，様相不動点論理 $R\mu$ を定義する．この論理は様相μ計算の一つのフラグメントに対応する．このフラグメントは CTL 論理式から変換できるものをすべて含んでおり，十分表現力があるといえる．

論理の健全性と完全性はしばしば，論理式全体がある代数構造の自由代数をなすという事実から示される．しかし，様相不動点論理に対してそのような代数構造を見つけることは，一般に困難である．

我々の第一の貢献は，論理 $R\mu$ に対応する代数構造 $\mathbf{RMu}$ を定式化することである．その困難さを克服するために，我々は，集合ではなく局所順序圏（Hom 集合に半順序の入った圏）の上の代数構造を与える．$R\mu$ 内の理論のシグネチャーは局所順序圏 $\Sigma_{\mathbf{Atm}}$ とみなされる．その理論の解釈は，$\Sigma_{\mathbf{Atm}}$ から $\mathbf{RMu}$ 代数への局所順序函手である．$\mathbf{RMu}$ 代数は，$R\mu$ を解釈するのに十分な構造の入った局所順序圏である． 自由 $\mathbf{RMu}$ 代数の構成は，この解釈のクラスに対して健全性と完全性を与える．

第二の貢献は，論理式保存定理が健全性や完全性の拡張になるような，２圏的な枠組みを与えることである．我々は，抽象化関係を，局所順序圏と局所順序函手からなるある２圏の２セル，すなわち，各コンポーネントが左随伴を持つような緩変換とみなす．この枠組みでは，論理式保存定理は，健全性や完全性と同様に２圏的な自由 $\mathbf{RMu}$ 代数の構成から導かれる．

我々の抽象化の定式化により，模倣や双模倣などのよく知られたさまざまな抽象化だけでなく，Kripke 構造からほかの構造への新しい抽象化もまた統一的に正当化される．

我々は，$R\mu$ の複雑な解釈を，より単純な解釈から構成する手法を，代数的アプローチで定式化する．解釈の意味論は，ファイブレーションのための函手の上の代数構造に基づく．その２圏的な自由代数の構成は，解釈の構成だけでなく，抽象化の構成も与える．

この論文は，一般化された Lawvere 理論として代数構造を定式化し，その自由代数の統一的な構成を与える．上に挙げた二つの２圏的な自由代数の構成は，その例として与えられる．

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Contents

Chapter 1

Introduction

This thesis formulates a unified notion of abstractions as they appear in abstract interpretations and abstract models for model checking. Our aim is to clarify the relationship between the soundness and completeness theorems for interpretations of logics and their formula-preservation theorems. We mean by the latter that the truth of formulas under interpretations is preserved across abstractions.

Formal methods have been used in order to discover bugs of hardware or software. Especially, model checking is the most successful as the technique to automatically verify transition systems modelling hardware and software systems against temporal formulas expressing specifications. However, direct verification of large systems is often infeasible. Abstraction is the method to formulate and extract from concrete transition systems only that part of information necessary for verification.

Abstraction has been studied in the context of various temporal logics [41, 29, 6, 49] and modal μ-calculus [26]. Various mathematical notions have been proposed as theoretical frameworks of abstraction: simulations [33], abstract interpretation [9, 34], and refinement [10, 22, 25]. They discuss which mathematical notion preserves formulas in which logic [31, 47, 48]. This thesis defines abstraction in a general setting that covers much of the above.

The requirements that guide our formulation of abstractions are the following.

- R1 The notion of abstractions is to be defined with respect to an explicitly specified logic.
- R2 An abstraction is a relation between two interpretations of the logic. We call

the two interpretations, concrete interpretation and abstract interpretation.

- R3 The formula-preservation theorem: whenever a formula is satisfied by the abstract interpretation, it is satisfied by the concrete one.

- R4 The formula-preservation theorem is to be proven compositionally from preservation of primitive formulas. We should be able to check whether a given relation is an abstraction or not just by checking primitive formulas.

- R5 The formula-preservation theorem should be derivable as a natural extension of soundness and completeness.

R5 is the concrete relationship we try to give between the soundness and completeness theorems and the formula-preservation theorem.

An abstraction satisfying these properties helps to check that a formula is true in a concrete interpretation, called satisfaction checking. By the formula-preservation theorem, the satisfaction checking is reduced to the following subproblems:

- P1 construction of an abstract interpretation,

- P2 construction of a relation between the concrete interpretation and the abstract interpretation,

- P3 proving that the relation is an abstraction, and

- P4 checking that the abstract interpretation satisfies the formula to be verified.

Since the subproblems P1 through P3 involve a fixed set of primitive formulas only, they can be solved once and for all in advance. For each formula to be verified, we only need to solve P4.

We remark that the requirements we made are not necessarily sufficient for the usefulness of abstractions. Satisfaction checking with respect to an abstract interpretation should be easier than that for the corresponding concrete interpretation. In order to rigorously discuss that, we need to formulate the hardness of satisfaction checking in general which is beyond the scope of this thesis. We instead illustrate our points through examples where it is intuitively clear that abstract satisfaction checking is indeed easier than concrete one.

We also discuss a solution of the subproblems P1 through P3. In applications, such as model checking of transition systems, it is important to construct an abstraction and an abstract interpretation from a concrete interpretation, semiautomatically.

Until now, abstract interpretations have been developed, in the case where a Kripke model is given as a concrete interpretation [7, 13]. Many of them assume a Kripke model as the abstract interpretation. However, does the abstract interpretation have to be a Kripke model? Without formulation of properties required for abstractions, we can not answer the question. In this thesis, however, we give properties R1 through R5 required for abstractions. Under the properties, we formulate abstractions in as general setting as possible and give a solution of the subproblems P1 through P3.

This thesis has three topics: modal fixed point logic $R\mu$, GLTS-formulas, and Lawvere A-theories.

1.1 Modal Fixed Point Logic $R\mu$

For the requirement R1, we define modal fixed point logic $R\mu$ in Chapter 2.

Our formulation is based on an algebraic approach. This chapter shows that our algebraic approach can be applied to a reasonably large fragment $L\mu^-$ of modal μ-calculus containing all of CTL translations. It is formulated in terms of a modal fixed point logic $R\mu$ into which the fragment can be translated. A Kleene algebra [28, 27] hints at fixed points operators of $R\mu$.

This chapter has two contributions. One is to give the logic $R\mu$ and the algebraic structure that corresponds to $R\mu$. The other is our 2-categorical formulation where $R\mu$ satisfies the requirements R2 through R5.

The first contribution has the following background. Algebraic approach helps us to uniformly prove soundness and completeness of logics. In this approach, one gives an algebraic structure T_L to a logic L, and L is interpreted in T_L-algebras. Here the important step is to find such T_L for which one can show that the syntax and the formal system of L form a free T_L-algebra; the soundness and the completeness are immediate consequences. For example, it is known that Boolean algebra, Heyting algebra, and modal algebra [45] correspond to classical propositional logic, intuitionistic propositional logic, and propositional modal logic K, respectively.

However, it is proved that modal μ-calculus corresponds to no algebraic structure on sets which is equationally finitely based [46]. Although a Kleene algebra is an essentially algebraic structure with least fixed points of only certain monotone functions, it does not have structures of lattices and greatest fixed points. An action algebra [43] which is given by adding an adjunction to a Kleene algebra, is known as an algebraic structure. Similarly to a Kleene algebra, however, it does not have structures of lattices and greatest fixed points. Therefore, it is not trivial to discover algebraic structures corresponding to modal fixed point logics.

We show that $R\mu$ corresponds to an algebraic structure not on the category of sets, but on the category of *locally ordered categories* (categories with partially ordered hom-sets). We call the algebraic structure **RMu**. The signature Σ of a theory in $R\mu$ is seen as a locally ordered category. Interpretations of the theory are certain locally ordered functors from Σ to an **RMu**-*algebra*, which is a locally ordered category with enough structure to interpret $R\mu$. We prove the theorem that the syntax and the formal system of $R\mu$ form a free **RMu**-algebra. The theorem implies soundness and completeness for this class of interpretations.

The second contribution of Chapter 2 is based on formulating abstraction as a *2-cell* of the following 2-category. The category of locally ordered categories extends to a 2-category by adding as 2-cells lax transformations whose components have left adjoints. We prove the theorem that the syntax and the formal system of $R\mu$ also form a free **RMu**-algebra on the 2-category. This theorem implies not only soundness and completeness, but also the formula-preservation theorem. We regard that the existence of the free **RMu**-algebra answers the requirement R5.

Our formulation of abstraction has two nice properties. The formula-preservation theorem uniformly justifies not only varieties of well-known abstractions, for example, simulations and bisimulations, but also a new abstraction from a Kripke structure to another structure. Moreover, we give a sufficient condition to construct an abstraction and an abstract interpretation from a concrete interpretation.

Chapter 2 has two contributions for this topic. One is the 2-categorical formulation of abstraction used to give the relationship between soundness (and completeness) of logics and the formula-preservation theorem. The other is to give the logic $R\mu$ and the algebraic structure **RMu** for the formulation.

1.2 GLTS-Formula

We formulate a construction method of a complex interpretation of $R\mu$ from a simpler one in algebraic approach. Extending the formulation to the 2-categorical one, we also give a construction method of abstractions.

In model checking, it is difficult to directly give abstractions of complex systems. So, they should be given by abstractions of components of the systems. Abstract-data mapping [7] and predicate abstraction [13] can be considered as the method to give abstraction of programs from abstractions of primitive commands. We make clear what construction of programs preserves abstractions of primitive commands in general setting.

Programs are formulated as *GLTS(generalised labelled transition system)-formulas* and primitive commands are formulated as *labels*. Interpretations of labels are given by fibrations with structures, called **GLTS**-*algebras*. We prove the theorem that GLTS-formulas are contained by the free **GLTS**-algebra generated from labels. The theorem implies that interpretations of GLTS-formulas are determined by interpretations of labels. Extending the setting to the 2-categorical one, we prove that abstractions for GLTS-formulas are determined by abstractions for labels. We also give the translation from an interpretation of a GLTS-formula into an interpretation of $R\mu$. The translation preserves abstraction.

1.3 Lawvere A-Theory

This thesis shows the reason why there exist free **RMu**-algebras and free **GLTS**-algebras in the above two topics. The point is a generalised algebraic structure, i.e., a set of basic operations and equations among derived operations, in a categorical, generalised sense. We formulate a generalised algebraic structure as a *Lawvere A-theory*. The notion of Lawvere A-theory is introduced as an invariant presentation of algebraic structures corresponding to a monad on a category A under reasonable conditions. A model of a Lawvere A-theory is what corresponds to an algebra of a monad. We prove the theorem that **RMu**-algebras are models of a Lawvere A-theory **RMu**, with A being the category of locally ordered categories. Since a Lawvere A-theory has free models, the theorem allows us to construct a free **RMu**-algebra from an arbitrary locally ordered small category. The formal system $R\mu$ gives a particular construction. Similarly, we prove the theorem that **GLTS**-algebras are models of a Lawvere A-theory **GLTS**, with A being the category of functors.

1.4 Organisation

In Chapter 2, we define modal fixed point logic $R\mu$. In Section 2.1, the syntax and the formal system of $R\mu$ are defined. We define the notion of **RMu**-algebra in Section 2.2 and define Δ-interpretations by using **RMu**-algebras in Section 2.3. The soundness and the completeness are proved by free algebra construction. We extend **RMu** to the 2-categorical one in Section 2.4 and define the notion of abstraction between Δ-interpretations in Section 2.5. The formula-preservation theorem is also proved by free algebra construction. In Section 2.6, $R\mu$ is compared with modal μ-calculus and CTL. In Section 2.7, an **RMu**-algebra is compared with other algebraic structures.

In Chapter 3, we define GLTS-formulas. In Section 3.1, the syntax of GLTS-formulas is defined. Section 3.2 explains cloven fibrations with structures. In Section 3.3, we define interpretations of labels and interpretations of GLTS-formulas. They are based on the notion of **GLTS**-algebras. We define the notion of abstraction between the interpretations in Section 3.4. In Section 3.5, we give the translation from an interpretation of a GLTS-formula into an interpretation of $R\mu$. In Section 3.6, various simulations are compared with the abstraction notion.

In Chapter 4, we define the notion of Lawvere A-theories, and show some algebraic structures as examples. In Section 4.1, we introduce our definition of Lawvere A-theory and the V-category of models of a theory. In Section 4.2, we show, in general, how to recover a Lawvere A-theory from its V-category of models: this gives a construction of a finitary V-monad on A from a Lawvere A-theory and shows that the definition of Lawvere A-theory is invariant with respect to its V-category of models. In Section 4.3, we start with a finitary V-monad on A, construct a Lawvere A-theory from it, and show how to recover the V-monad. Combining this with the work of Section 4.2 yields the correspondence we seek between Lawvere A-theories and finitary V-monads on A. In Section 4.4, we analyse examples where A is $\mathbf{Cat}_o$ or $\mathbf{Cat}$. We define Lawvere A-theory **RMu** in Section 4.5 and **GLTS** in Section 4.6.

Chapter 2

Modal Fixed Point Logic $R\mu$

This section gives the logic $R\mu$ and the algebraic structure that corresponds to $R\mu$. For the logic, we formulate abstraction such that the formula-preservation theorem becomes an extension of soundness and completeness theorems.

2.1 Syntax and Formal System $R\mu$

This section defines the formal system $R\mu$. Our motivation for $R\mu$ is to give a subclass of modal μ-calculus $L\mu$ [26] for which we can prove the soundness and the completeness by free algebra construction.

The negation normal form of $L\mu$-formulas is given by the following grammar.

$$\begin{array}{llllllll} \varphi & ::= & p & \mid \neg p & \mid \Diamond\varphi & \mid \Box\varphi & \mid Z & \\ & \mid & \bot & \mid \top & \mid \varphi \vee \varphi & \mid \varphi \wedge \varphi & \mid \mu Z.\varphi & \mid \nu Z.\varphi \end{array}$$

A completeness result of $R\mu$ for the class of Kripke models is given by Walukiewicz [51], but the proof is not based on free algebra construction. We would like to find a suitable algebraic structure for the set of $L\mu$-formulas, but doing so for the whole of $L\mu$ is made difficult due to the fixed-point operators only with the positivity condition (but see Section 5.2). The problem is avoided here by restricting the logic and by considering algebraic structures on categories on sets.

Kleene algebra [28, 27] hints at our solution to restrict $L\mu$-formulas. It has the least fixed point $a^* \cdot b$ of the following function.

$$x \mapsto b + a \cdot x$$

We introduce a similar least fixed point operator which takes two parameters in our system $R\mu$. Propositional variables are not used in $R\mu$. Instead of substitution, $R\mu$ has a composition of two $R\mu$-formulas.

The syntax of $R\mu$ is parametrised by a *signature* **Atm** which is a set of atomic formulas. Henceforth we fix an arbitrary signature. *$R\mu$-formulas* are given by the following grammar ($a \in \mathbf{Atm}$).

$$\begin{array}{rcl} \varphi & ::= & a \mid \mathbf{Id} \mid \varphi \circ \varphi \\ & & \mid \bot \mid \top \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \mu(\varphi, \varphi) \mid \nu(\varphi, \varphi) \end{array}$$

Unlike modal logic, $R\mu$ does not syntactically distinguish a proposition and a modal operator; **Atm** may contain both. Also, it does not have negation nor demand the distributivity between $\vee$ and $\wedge$. These are added later when necessary. This is in order to define the notion of abstraction which includes as many examples as possible, for example, not only bisimulations but also simulations (see Section 2.5).

The inference rules of $R\mu$ derives judgements, or *entailments*, of the form $\varphi \Rightarrow \psi$. The form $\varphi = \psi$ abbreviates the conjunction of $\psi \Rightarrow \varphi$ and $\varphi \Rightarrow \psi$. A *theory* Δ in $R\mu$ is a set of entailments treated as axioms. The judgements derivable from Δ are Δ-*theorems*.

Partial Order

$$\frac{}{\varphi \Rightarrow \varphi} \qquad \frac{\varphi \Rightarrow \psi \quad \psi \Rightarrow \sigma}{\varphi \Rightarrow \sigma}$$

Bounded Lattice The set of all $R\mu$-formulas is a bounded lattice. Note that it is not always a distributive lattice.

$$\frac{}{\bot \Rightarrow \psi} \quad \frac{\varphi \Rightarrow \sigma \quad \psi \Rightarrow \sigma}{\varphi \vee \psi \Rightarrow \sigma} \quad \frac{}{\varphi \Rightarrow \varphi \vee \psi} \quad \frac{}{\psi \Rightarrow \varphi \vee \psi}$$

$$\frac{}{\psi \Rightarrow \top} \quad \frac{\sigma \Rightarrow \varphi \quad \sigma \Rightarrow \psi}{\sigma \Rightarrow \varphi \wedge \psi} \quad \frac{}{\varphi \wedge \psi \Rightarrow \varphi} \quad \frac{}{\varphi \wedge \psi \Rightarrow \psi}$$

Composition Intuitively, an $R\mu$-formula is a formula with at most one free propositional variable. Composition here corresponds to substitution to that variable. See Section 2.6 for the translation between the two formulations.

$$\frac{\varphi \Rightarrow \sigma \quad \psi \Rightarrow \tau}{\varphi \circ \psi \Rightarrow \sigma \circ \tau} \quad \frac{}{(\varphi \circ \psi) \circ \sigma = \varphi \circ (\psi \circ \sigma)}$$

$$\overline{\mathbf{Id} \circ \varphi = \varphi} \quad \overline{\varphi \circ \mathbf{Id} = \varphi}$$

Parameterised Fixed Point of Restricted Formula These rules specify that $\mu(\varphi, \psi)$ is the least fixed point of F where $F\sigma \triangleq \varphi \vee (\psi \circ \sigma)$ and $\nu(\varphi, \psi)$ is the greatest fixed point of G where $G\sigma \triangleq \varphi \wedge (\psi \circ \sigma)$.

$$\frac{}{\varphi \Rightarrow \mu(\varphi, \psi)} \quad \frac{}{\psi \circ \mu(\varphi, \psi) \Rightarrow \mu(\varphi, \psi)} \quad \frac{\varphi \Rightarrow \sigma \quad \psi \circ \sigma \Rightarrow \sigma}{\mu(\varphi, \psi) \Rightarrow \sigma}$$

$$\frac{}{\nu(\varphi, \psi) \Rightarrow \varphi} \quad \frac{}{\nu(\varphi, \psi) \Rightarrow \psi \circ \nu(\varphi, \psi)} \quad \frac{\sigma \Rightarrow \varphi \quad \sigma \Rightarrow \psi \circ \sigma}{\sigma \Rightarrow \nu(\varphi, \psi)}$$

Right Distributivity Right composition of formulas preserves the structure of bounded lattices and fixed points.

$$\overline{\bot \circ \psi \Rightarrow \bot} \quad \overline{(\varphi \vee \psi) \circ \sigma \Rightarrow (\varphi \circ \sigma) \vee (\psi \circ \sigma)}$$

$$\overline{\top \Rightarrow \top \circ \psi} \quad \overline{(\varphi \circ \sigma) \wedge (\psi \circ \sigma) \Rightarrow (\varphi \wedge \psi) \circ \sigma}$$

$$\overline{\mu(\varphi, \psi) \circ \sigma \Rightarrow \mu(\varphi \circ \sigma, \psi)} \quad \overline{\nu(\varphi \circ \sigma, \psi) \Rightarrow \nu(\varphi, \psi) \circ \sigma}$$

2.2 Algebraic Structure RMu on the Category of Locally Ordered Categories

To give an algebraic semantics for $R\mu$, we now define the structure we call **RMu**-algebras on *locally ordered small categories.*

A locally ordered small category is a small category whose homsets are equipped with partial orders and the composition is order-preserving. Typical examples include the category of sets and relations, and that of partially ordered sets and monotone functions. Our main example is the full subcategory $\mathbf{Pos}_{\mathbf{CL}}$ of the latter, that is: objects are complete lattices[1]; arrows are all monotone functions; orders are given pointwise. We see in Section 2.3 that $R\mu$-formulas also give rise to a locally ordered category with formulas as arrows and with just one formal object.

Definition 2.2.1. A locally ordered small category C is an **RMu**-*algebra* if

[1] In order to fit C in small locally ordered categories, we should limit the size of lattices. It is not a problem, but here we generally wave our hands on the size issue.

- for objects x and y, the hom-poset is a bounded lattice

$$(C(x, y), \leq_{x,y}, \vee_{x,y}, \wedge_{x,y}, \perp_{x,y}, \top_{x,y});$$

- for objects x, y, z, and arrow $f\colon x \to y$, the pre-composition function $C(f, z)\colon C(y, z) \to C(x, z)$ preserves the structure of the bounded lattice;

- for arrows $f\colon x \to y$ and $g\colon y \to y$, $C(x, y)$ has a least fixed point $\mu_{x,y}(f, g)$ of the monotone function which sends $\sigma\colon x \to y$ to $f \vee_{x,y} (g \circ \sigma)\colon x \to y$;

- for arrows $f\colon x \to y$ and $g\colon y \to y$, $C(x, y)$ has a greatest fixed point $\nu_{x,y}(f, g)$ of the monotone function which sends $\sigma\colon x \to y$ to $f \wedge_{x,y} (g \circ \sigma)\colon x \to y$; and

- for arrows $h\colon x \to y$, $f\colon y \to z$, and $g\colon z \to z$, $\mu_{y,z}(f, g) \circ h = \mu_{x,z}(f \circ h, g)$ and $\nu_{y,z}(f, g) \circ h = \nu_{x,z}(f \circ h, g)$.

The category $\mathbf{Pos_{CL}}$ is also an **RMu**-algebra.

- Bounded lattice structures of hom-posets are given by least element, greatest element, join, and meet of complete lattices, respectively.

- For arrows $f\colon X \to Y$ and $g\colon Y \to Y$, $\mu_{X,Y}(f, g)$ is the function which sends $x \in X$ to $\wedge_Y\{y \in Y \mid f(x) \vee_Y g(y) \leq_Y y\}$.

- For arrows $f\colon X \to Y$ and $g\colon Y \to Y$, $\nu_{X,Y}(f, g)$ is the function which sends $x \in X$ to $\vee_Y\{y \in Y \mid y \leq_Y f(x) \wedge_Y g(y)\}$.

The point of **RMu**-algebras is that they are models of a Lawvere A-theory, **RMu**, in the sense of Chapter 4, with A being the category of locally ordered categories. A Lawvere A-theory is a formulation of an algebraic structure, i.e., a set of basic operations and equations among derived operations, in a categorical, generalised sense. This allows us to automatically deduce our results from the general theory of algebraic structures. We can give the Lawvere A-theory **RMu** for **RMu**-algebras. Let **LocOrd** be the category whose objects are locally ordered small categories and whose arrows are locally ordered functors (i.e., functors that preserve the order).

Theorem 2.2.2 (See Section 4.5.1)**.** There exists a Lawvere **LocOrd**-theory **RMu** such that the models are all **RMu**-algebras.

As with any Lawvere A-theory, the models of **RMu**, i.e., **RMu**-algebras, form a category. We write **RMu-Alg** for the category. Arrows of **RMu-Alg** are locally ordered functors which strictly preserve structures of **RMu**-algebras. That is to say, an arrow $H\colon C \to D$ of **RMu-Alg** is a locally ordered functor which satisfies the following equations for objects x and y of C.

$$\begin{array}{rcl} H\bot_{x,y} & = & \bot_{Hx,Hy} \\ H\top_{x,y} & = & \top_{Hx,Hy} \\ H(\varphi \vee_{x,y} \psi) & = & H\varphi \vee_{Hx,Hy} H\psi \\ H(\varphi \wedge_{x,y} \psi) & = & H\varphi \wedge_{Hx,Hy} H\psi \\ H(\mu_{x,y}(\varphi, \psi)) & = & \mu_{Hx,Hy}(H\varphi, H\psi) \\ H(\nu_{x,y}(\varphi, \psi)) & = & \nu_{Hx,Hy}(H\varphi, H\psi) \end{array}$$

The main fact we use from the general theory of Lawvere A-theory is that there is an adjunction between **RMu-Alg** and **LocOrd**.

Theorem 2.2.3 (See Section 4.5.1)**.** The forgetful functor U from **RMu-Alg** to **LocOrd** has a left adjoint.

So one can construct a free **RMu**-algebra from an arbitrary locally ordered small category. As we see in the next section, the algebraic semantics of $R\mu$, and its soundness and completeness, all hinge on the fact that $R\mu$ is a free **RMu**-algebra.

2.3 Algebraic Semantics of $R\mu$

For a theory Δ in $R\mu$, this section gives what it means to give an interpretation of it in an **RMu**-algebra. We prove soundness and completeness of the formal system $R\mu$ with respect to the class of Δ-interpretations.

We regard signature **Atm** as the graph with just one node $*$ and with elements of **Atm** as edges. Let $\Sigma_{\mathbf{Atm}}$ be the locally ordered small category freely generated from this graph. Then, to give a locally ordered functor from $\Sigma_{\mathbf{Atm}}$ to another locally ordered category C is equivalent to giving a function from **Atm** to $C(c, c)$ for some $c \in C$.

There exists a free **RMu**-algebra on $\Sigma_{\mathbf{Atm}}$ by Theorem 2.2.3. The formal system $R\mu$ gives a particular construction.

The syntactic entities of $R\mu$ can be organised into the **RMu**-algebra $F\Sigma_{\mathbf{Atm}}$ defined by

- objects: unique object $*$
- arrows: $R\mu$-formulas quotiented by $\emptyset$-theorem $\varphi = \psi$ (i.e., Δ-theorem $\varphi = \psi$ in the case where Δ is the empty set)
- inequality: $\emptyset$-theorem $\varphi \Rightarrow \psi$
- **RMu**-structures: constructors of $R\mu$-formulas.

We write $\eta\colon \Sigma_{\mathbf{Atm}} \to UF\Sigma_{\mathbf{Atm}}$ for the trivial inclusion. $F\Sigma_{\mathbf{Atm}}$ is a free **RMu**-algebra in the following sense.

Theorem 2.3.1 (Free algebra)**.** For **RMu**-algebra M and locally ordered functor $m\colon \Sigma_{\mathbf{Atm}} \to UM$, m is equal to $U[\![-]\!]_m \circ \eta$ for a unique arrow $[\![-]\!]_m\colon F\Sigma_{\mathbf{Atm}} \to M$ in **RMu-Alg**.

Proof. We only show the construction of $[\![-]\!]_m$. $[\![*]\!]_m$ is $m*$. The arrow part of $[\![-]\!]_m$ is defined inductively. That $[\![-]\!]_m$ is locally ordered is proved by induction on the structure of derivations. □

We can now talk about whether an entailment holds in an **RMu**-algebra, using $[\![-]\!]_m$.

Definition 2.3.2 (Δ-interpretation)**.** For an **RMu**-algebra M and a theory Δ, a locally ordered functor $m\colon \Sigma_{\mathbf{Atm}} \to UM$ is a Δ-*interpretation* if $[\![\phi]\!]_m \leq_m [\![\psi]\!]_m$ for each axiom $\phi \Rightarrow \psi$ in Δ.

Examples below are Δ-interpretations where Δ is the empty set. For examples with non-empty Δ, see Section 2.6.

Example 2.3.3 (Transition system)**.** Let **Atm** be $\{\Diamond, \Box\}$. A transition system $(S,\, R \subseteq S \times S)$ gives rise to the $\emptyset$-interpretation $m\colon \Sigma_{\mathbf{Atm}} \to U\mathbf{Pos}_{\mathbf{CL}}$ given by

$m* = \wp(S)$

$m\Diamond\colon\ X \mapsto \{s \in S \mid \exists s' \in X.(s, s') \in R\}$

$m\Box\colon\ X \mapsto \{s \in S \mid \forall s' \in S.(s, s') \in R \Rightarrow s' \in X\}$

Since the definition of Δ-interpretations does not require the duality of $\Diamond$ and $\Box$, the following is also an example of $\emptyset$-interpretation.

Example 2.3.4 (Modal transition system)**.** A modal transition system [30] is a pair of two transition systems $(S,\ R_{\text{must}} \subseteq S \times S)$ and $(S,\ R_{\text{may}} \subseteq S \times S)$ such that $R_{\text{must}} \subseteq R_{\text{may}}$. Let **Atm** be $\{\Diamond, \Box\}$. A modal transition system gives rise to the $\emptyset$-interpretation $m\colon\ \Sigma_{\mathbf{Atm}} \to U\mathbf{Pos}_{\mathbf{CL}}$ by

$m* = \wp(S)$

$m\Diamond\colon\ X \mapsto \{s \in S \mid \exists s' \in X.(s, s') \in R_{\text{must}}\}$

$m\Box\colon\ X \mapsto \{s \in S \mid \forall s' \in S.(s, s') \in R_{\text{may}} \Rightarrow s' \in X\}$

A Kripke structure is also an example of $\emptyset$-interpretation.

Example 2.3.5 (Kripke semantics)**.** Let P be a set and **Atm** be $P \cup \{\neg p \mid p \in P\} \cup \{\Diamond, \Box\}$. A Kripke structure $(S,\ R \subseteq S \times S,\ Q\colon\ S \to \wp(P))$ gives rise to the $\emptyset$-interpretation $m\colon\ \Sigma_{\mathbf{Atm}} \to U\mathbf{Pos}_{\mathbf{CL}}$ given by

$m* = \wp(S)$

$mp\colon\ X \mapsto \{s \in S \mid p \in Q(s)\}$ for $p \in P$

$m\neg p\colon\ X \mapsto \{s \in S \mid p \notin Q(s)\}$ for $p \in P$

$m\Diamond\colon\ X \mapsto \{s \in S \mid \exists s' \in X.(s, s') \in R\}$

$m\Box\colon\ X \mapsto \{s \in S \mid \forall s' \in S.(s, s') \in R \Rightarrow s' \in X\}$

This interpretation m sends atomic propositions to constant functions and sends $\neg p$ to the complement of mp. However, note that the definition of Δ-interpretations does not require such conditions. The next example is not boolean.

Example 2.3.6. Let $(\mathbf{Z} \cup \{-\infty, \infty\}, \leq_{\mathbf{Z}})$ be the partially ordered set of all integers, the least element $-\infty$, and the greatest element ∞. Let **Intvl** be the following set. Elements of the set represent intervals in $\mathbf{Z} \cup \{-\infty, \infty\}$.

$$\{(a, b) \mid a \in (\mathbf{Z} \cup \{-\infty\}), b \in (\mathbf{Z} \cup \{\infty\}), a \leq_{\mathbf{Z}} b\} \cup \{\bot\}$$

We define the order $i \leq_{\mathbf{Intvl}} i'$ on **Intvl** by

- $i = \bot$, or
- $i = (a, b)$, $i' = (a', b')$, $a' \leq_{\mathbf{Z}} a$, and $b \leq_{\mathbf{Z}} b'$.

This set is a complete lattice. When **Atm** is a singleton set, a monotone function on **Intvl** gives rise to a $\emptyset$-interpretation $m\colon\ \Sigma_{\mathbf{Atm}} \to U\mathbf{Pos}_{\mathbf{CL}}$.

Theorem 2.3.1 implies both the soundness and completeness of $R\mu$ for this semantics only when $\Delta = \emptyset$. For the general case, the soundness is still immediate, by induction on the structure of derivations.

Theorem 2.3.7 (Soundness)**.** For Δ-interpretation m, if $\varphi \Rightarrow \psi$ is a Δ-theorem, then $[\![\varphi]\!]_m \leq_{m*,m*} [\![\psi]\!]_m$.

The completeness is proven by the construction of a generic model [39].

Theorem 2.3.8 (Completeness)**.** For $R\mu$-formulas φ and ψ, the judgement $\varphi \Rightarrow \psi$ is a Δ-theorem if every Δ-interpretation m satisfies $[\![\varphi]\!]_m \leq_{m*,m*} [\![\psi]\!]_m$.

Proof. We give an **RMu**-algebra $F\Sigma_{\mathbf{Atm}}/\Delta$:

- objects: only $*$
- arrows: $R\mu$-formulas quotiented by Δ-theorems $\varphi = \psi$
- inequality: Δ-theorems $\varphi \Rightarrow \psi$.

The trivial embedding $\eta_\Delta : \Sigma_{\mathbf{Atm}} \to U(F\Sigma_{\mathbf{Atm}}/\Delta)$ becomes a Δ-interpretation. We have that $\varphi \Rightarrow \psi$ is a Δ-theorem whenever $[\![\varphi]\!]_{\eta_\Delta} \leq_{\eta_\Delta *, \eta_\Delta *} [\![\psi]\!]_{\eta_\Delta}$. □

2.4 Algebraic Structure on the 2-Category of Locally Ordered Categories

We use the notion of 2-category to uniformly extend the analysis of the previous sections, enriching the set of interpretations $\mathbf{LocOrd}(\Sigma_{\mathbf{Atm}}, UM)$ to a category having interpretations as objects. Following [25], we model abstractions as certain lax transformations. Given two locally ordered functors $m, n\colon C \to D$, a *lax transformation* from m to n is an $\mathbf{ob}(C)$-indexed family γ of arrows in D, which satisfies the following condition. For each $c \in \mathbf{ob}(C)$, $\gamma_c \in D(mc, nc)$; moreover, for each arrow $f \in c \to c'$ of C, $nf \circ \gamma_c \leq_{mc,nc'} \gamma_{c'} \circ mf$.

Definition 2.4.1. $\mathbf{LocOrd}_{lr}$ is the 2-category given by

- objects: locally ordered small categories
- arrows: locally ordered functors
- 2-cells: lax transformations whose components have left adjoints (i.e., lax transformation $\gamma\colon m \to n$ such that each component $\gamma_x\colon mx \to nx$ has a left adjoint; namely, there exists an $\alpha_x\colon nx \to mx$ such that $\alpha_x \circ \gamma_x \leq \mathbf{Id}_{mx}$ and $\mathbf{Id}_{nx} \leq \gamma_x \circ \alpha_x$).

The 2-category $\mathbf{RMu\text{-}Alg}_{lr}$ is defined similarly: objects and arrows are those of $\mathbf{RMu\text{-}Alg}$, and 2-cells are given as above.

Now, Theorem 2.3.1 can be extended in the 2-categorical context.

Theorem 2.4.2 (Free algebra)**.** For $\mathbf{RMu}$-algebra M, there exists an isomorphism between the category $\mathbf{RMu\text{-}Alg}_{lr}(F\Sigma_{\mathbf{Atm}}, M)$ and the category $\mathbf{LocOrd}_{lr}(\Sigma_{\mathbf{Atm}}, UM)$.

Proof. The bijection on objects is given by Theorem 2.3.1. The bijection on arrows is given as follows. An arrow of $\mathbf{RMu\text{-}Alg}_{lr}(F\Sigma_{\mathbf{Atm}}, M)$ is always an arrow of $\mathbf{LocOrd}_{lr}(\Sigma_{\mathbf{Atm}}, UM)$, since $F\Sigma_{\mathbf{Atm}}$ has the same objects as $\Sigma_{\mathbf{Atm}}$ and includes all $\Sigma_{\mathbf{Atm}}$ arrows. So it suffices to show that an arrow $\gamma\colon m \to n$ of $\mathbf{LocOrd}_{lr}(\Sigma_{\mathbf{Atm}}, UM)$ is always an arrow of $\mathbf{RMu\text{-}Alg}_{lr}(F\Sigma_{\mathbf{Atm}}, M)$. Let α_* be the left adjoint of γ_*. By induction on the structure of $R\mu$-formula φ, we prove that γ_* gives a lax transformation from $[\![-]\!]_m$ to $[\![-]\!]_n$. (i.e., $[\![\varphi]\!]_n \circ \gamma_* \leq \gamma_* \circ [\![\varphi]\!]_m$ for each arrow $\varphi\colon * \to *$ of $F\Sigma_{\mathbf{Atm}}$.)

Case 1.

$$\begin{aligned}
[\![a]\!]_n \circ \gamma_* &= na \circ \gamma_* \\
&\leq \gamma_* \circ ma \\
&= \gamma_* \circ [\![a]\!]_m
\end{aligned}$$

Case 2.

$$\begin{aligned}
[\![\mathbf{Id}]\!]_n \circ \gamma_* &= \mathbf{Id} \circ \gamma_* \\
&= \gamma_* \\
&= \gamma_* \circ \mathbf{Id} \\
&= \gamma_* \circ [\![\mathbf{Id}]\!]_m
\end{aligned}$$

Case 3.

$$\begin{aligned}
[\![\varphi \circ \psi]\!]_n \circ \gamma_* &= [\![\varphi]\!]_n \circ [\![\psi]\!]_n \circ \gamma_* \\
&\leq [\![\varphi]\!]_n \circ \gamma_* \circ [\![\psi]\!]_m \\
&\leq \gamma_* \circ [\![\varphi]\!]_m \circ [\![\psi]\!]_m \\
&= \gamma_* \circ [\![\varphi \circ \psi]\!]_m
\end{aligned}$$

Case 4.

$$\begin{aligned}
[\![\bot]\!]_n \circ \gamma_* &= \bot_{n*,n*} \circ \gamma_* \\
&= \bot_{m*,n*} \\
&\leq \gamma_* \circ [\![\bot]\!]_m
\end{aligned}$$

Case 5.

$$\begin{aligned}
[\![\top]\!]_n \circ \gamma_* &\leq \gamma_* \circ \alpha_* \circ [\![\top]\!]_n \circ \gamma_* \\
&\leq \gamma_* \circ \top_{m*,m*} \\
&= \gamma_* \circ [\![\top]\!]_m
\end{aligned}$$

Case 6.

$$\begin{aligned}
[\![\varphi \vee \psi]\!]_n \circ \gamma_* &= ([\![\varphi]\!]_n \vee_{n*,n*} [\![\psi]\!]_n) \circ \gamma_* \\
&= ([\![\varphi]\!]_n \circ \gamma_*) \vee_{m*,n*} ([\![\psi]\!]_n \circ \gamma_*) \\
&\leq (\gamma_* \circ [\![\varphi]\!]_m) \vee_{m*,n*} (\gamma_* \circ [\![\psi]\!]_m) \\
&\leq \gamma_* \circ ([\![\varphi]\!]_m \vee_{m*,m*} [\![\psi]\!]_m) \\
&= \gamma_* \circ [\![\varphi \vee \psi]\!]_m
\end{aligned}$$

Case 7.

$$\begin{aligned}
& [\![\varphi \wedge \psi]\!]_n \circ \gamma_* \leq \gamma_* \circ [\![\varphi \wedge \psi]\!]_m \\
\iff\ & \alpha_* \circ [\![\varphi \wedge \psi]\!]_n \circ \gamma_* \leq [\![\varphi \wedge \psi]\!]_m \\
\iff\ & \alpha_* \circ [\![\varphi \wedge \psi]\!]_n \circ \gamma_* \leq [\![\varphi]\!]_m \wedge_{m*,m*} [\![\psi]\!]_m \\
\iff\ & \alpha_* \circ [\![\varphi \wedge \psi]\!]_n \circ \gamma_* \leq [\![\varphi]\!]_m \qquad \text{and} \\
& \alpha_* \circ [\![\varphi \wedge \psi]\!]_n \circ \gamma_* \leq [\![\psi]\!]_m \\
\iff\ & [\![\varphi \wedge \psi]\!]_n \circ \gamma_* \leq \gamma_* \circ [\![\varphi]\!]_m \qquad \text{and} \\
& [\![\varphi \wedge \psi]\!]_n \circ \gamma_* \leq \gamma_* \circ [\![\psi]\!]_m \\
\iff\ & ([\![\varphi]\!]_n \wedge_{n*,n*} [\![\psi]\!]_n) \circ \gamma_* \leq \gamma_* \circ [\![\varphi]\!]_m \qquad \text{and} \\
& ([\![\varphi]\!]_n \wedge_{n*,n*} [\![\psi]\!]_n) \circ \gamma_* \leq \gamma_* \circ [\![\psi]\!]_m \\
\Longleftarrow\ & [\![\varphi]\!]_n \circ \gamma_* \leq \gamma_* \circ [\![\varphi]\!]_m \qquad \text{and} \\
& [\![\psi]\!]_n \circ \gamma_* \leq \gamma_* \circ [\![\psi]\!]_m
\end{aligned}$$

Case 8.

$$\begin{aligned}
& [\![\mu(\varphi, \psi)]\!]_n \circ \gamma_* \leq \gamma_* \circ [\![\mu(\varphi, \psi)]\!]_m \\
\iff\ & \mu_{n*,n*}([\![\varphi]\!]_n, [\![\psi]\!]_n) \circ \gamma_* \leq \gamma_* \circ [\![\mu(\varphi, \psi)]\!]_m \\
\iff\ & \mu_{m*,n*}([\![\varphi]\!]_n \circ \gamma_*, [\![\psi]\!]_n) \leq \gamma_* \circ [\![\mu(\varphi, \psi)]\!]_m \\
\Longleftarrow\ & [\![\varphi]\!]_n \circ \gamma_* \leq \gamma_* \circ \mu_{m*,m*}([\![\varphi]\!]_m, [\![\psi]\!]_m) \qquad \text{and} \\
& [\![\psi]\!]_n \circ \gamma_* \circ \mu_{m*,m*}([\![\varphi]\!]_m, [\![\psi]\!]_m) \\
& \leq \gamma_* \circ \mu_{m*,m*}([\![\varphi]\!]_m, [\![\psi]\!]_m) \\
\Longleftarrow\ & \gamma_* \circ [\![\varphi]\!]_m \leq \gamma_* \circ \mu_{m*,m*}([\![\varphi]\!]_m, [\![\psi]\!]_m) \qquad \text{and} \\
& \gamma_* \circ [\![\psi]\!]_m \circ \mu_{m*,m*}([\![\varphi]\!]_m, [\![\psi]\!]_m) \\
& \leq \gamma_* \circ \mu_{m*,m*}([\![\varphi]\!]_m, [\![\psi]\!]_m)
\end{aligned}$$

Case 9.

$$
\begin{aligned}
& [\![\nu(\varphi,\psi)]\!]_n \circ \gamma_* \le \gamma_* \circ [\![\nu(\varphi,\psi)]\!]_m \\
\iff\ & \alpha_* \circ [\![\nu(\varphi,\psi)]\!]_n \circ \gamma_* \le [\![\nu(\varphi,\psi)]\!]_m \\
\iff\ & \alpha_* \circ [\![\nu(\varphi,\psi)]\!]_n \circ \gamma_* \le \nu([\![\varphi]\!]_m, [\![\psi]\!]_m) \\
\Longleftarrow\ & \alpha_* \circ [\![\nu(\varphi,\psi)]\!]_n \circ \gamma_* \le [\![\varphi]\!]_m \qquad \text{and} \\
& \alpha_* \circ [\![\nu(\varphi,\psi)]\!]_n \circ \gamma_* \\
& \le [\![\psi]\!]_m \circ \alpha_* \circ [\![\nu(\varphi,\psi)]\!]_n \circ \gamma_* \\
\iff\ & \alpha_* \circ \nu([\![\varphi]\!]_n, [\![\psi]\!]_n)_{n*,n*} \circ \gamma_* \le [\![\varphi]\!]_m \qquad \text{and} \\
& \alpha_* \circ \nu_{n*,n*}([\![\varphi]\!]_n, [\![\psi]\!]_n) \circ \gamma_* \\
& \le [\![\psi]\!]_m \circ \alpha_* \circ \nu_{n*,n*}([\![\varphi]\!]_n, [\![\psi]\!]_n) \circ \gamma_* \\
\Longleftarrow\ & \alpha_* \circ [\![\varphi]\!]_n \circ \gamma_* \le [\![\varphi]\!]_m \qquad \text{and} \\
& \alpha_* \circ [\![\psi]\!]_n \circ \nu_{n*,n*}([\![\varphi]\!]_n, [\![\psi]\!]_n) \circ \gamma_* \\
& \le [\![\psi]\!]_m \circ \alpha_* \circ \nu_{n*,n*}([\![\varphi]\!]_n, [\![\psi]\!]_n) \circ \gamma_* \\
\Longleftarrow\ & \alpha_* \circ [\![\varphi]\!]_n \circ \gamma_* \le \alpha_* \circ \gamma_* \circ [\![\varphi]\!]_m \qquad \text{and} \\
& \alpha_* \circ [\![\psi]\!]_n \le [\![\psi]\!]_m \circ \alpha_* \\
\Longleftarrow\ & [\![\varphi]\!]_n \circ \gamma_* \le \gamma_* \circ [\![\varphi]\!]_m \qquad \text{and} \\
& [\![\psi]\!]_n \circ \gamma_* \le \gamma_* \circ [\![\psi]\!]_m
\end{aligned}
$$

□

In fact, there exists a general theory of 2-categorical Lawvere A-theories. There exists an enriched version of **RMu** as a Lawvere $\mathbf{LocOrd}_{lr}$-theory. The 2-category of its models can be shown to be $\mathbf{RMu\text{-}Alg}_{lr}$ by a discussion similar to the above proof. That the forgetful functor from $\mathbf{RMu\text{-}Alg}_{lr}$ to $\mathbf{LocOrd}_{lr}$ has a left 2-adjoint can also be shown in this way (See Section 4.5.2).

2.5 Abstraction between Interpretations

This section gives the notion of abstraction from a Δ-interpretation to another, which allows a general construction of abstract interpretations.

Abstract interpretation [9, 34] is a general framework to unify various static analysis of programs. In this framework, they give various semantics for each property to verify, for example, trace semantics, relational semantics, big-step operational semantics, small-step operational semantics, and collecting semantics. Correctness of relations among different semantics is justified by *Galois connections*.

On one hand, in abstract interpretation, verification of programs is not always based on modal fixed point logics. Properties to verify are not given by formula, but by variation of semantic domains. On the other hand, in *model checking*, verification of programs is based on modal fixed point logics, for example, LTL, CTL, CTL*, modal μ-calculus. In the framework of model checking, the space of properties to verify is determined by the logics. One can give various transition systems for each complexity of verification. Correctness of relations among different transition systems is justified by *simulations*.

This chapter unifies both frameworks. Properties to verify are given by $R\mu$-formula. Moreover, we can give various semantics other than transition systems. We can also give abstractions other than simulations.

Theorem 2.4.2 guides us to define the notion of abstraction. This is the most central definition in this chapter. By the following definition, one can prove the formula-preservation theorem as a corollary of Theorem 2.4.2.

Definition 2.5.1 (Abstraction)**.** An *abstraction* γ from a Δ-interpretation m to another n is a 2-cell $\gamma\colon m \to n$ in $\mathbf{LocOrd}_{lr}$.

Modal logics require the negations, the distributivity between $\vee$ and $\wedge$, duality of modal operators, and the difference between atomic propositions and modal operators. However, our definition of Δ-interpretations does not require them. This notion has many examples as follows.

Example 2.5.2 (Simulation)**.** Let $\mathbf{Atm}$ be $P \cup \{\neg p \mid p \in P\} \cup \{\Box\}$. Similarly to Example 2.3.5, let $\emptyset$-interpretation m correspond to a Kripke model $K = (S, R, Q)$ and n correspond to a Kripke model $K' = (S', R', Q')$. A relation $\rho \subseteq S \times S'$ gives rise to the following arrow $\gamma_*\colon m* \to n*$. It has the left adjoint α_*.

$$\gamma_*(X) = \{s' \in S' \mid \forall s \in S.(s, s') \in \rho \Rightarrow s \in X\}$$
$$\alpha_*(X) = \{s \in S \mid \exists s' \in X.(s, s') \in R\}$$

Then, γ_* gives rise to an abstraction γ from m to n if and only if K' simulates K with ρ.

Example 2.5.3 (Converse simulation)**.** Let $\mathbf{Atm}$ be $P \cup \{\neg p \mid p \in P\} \cup \{\Diamond\}$. Similarly to the previous example, a relation ρ and Kripke models K, K' give rise to the arrow $\gamma_* \colon m* \to n*$. Here, $m\Diamond$ and $n\Diamond$ are given similarly to Example 2.3.5. Then, γ_* gives rise to an abstraction γ from m to n if and only if K' simulates K with ρ.

Example 2.5.4 (Bisimulation)**.** Let $\mathbf{Atm}$ be $P \cup \{\neg p \mid p \in P\} \cup \{\Diamond, \Box\}$. Similarly to the previous examples, a relation ρ and Kripke models K, K' give rise to the arrow $\gamma_* \colon m* \to n*$. Then, γ_* gives rise to an abstraction γ from m to n if and only if ρ is a bisimulation between K and K'.

Example 2.5.5 (Abstraction of modal transition system)**.** An abstraction, in the sense of [30], of a modal transition system $K = ((S, R_{\mathrm{must}}), (S, R_{\mathrm{may}}))$ is a modal transition system $K' = ((S', R'_{\mathrm{must}}), (S', R'_{\mathrm{may}}))$ with a relation $\rho \subseteq S \times S'$ such that (S', R'_{may}) simulates (S, R_{may}) with ρ and (S, R_{must}) simulates (S', R'_{must}) with ρ. This relation preserves all formulas in modal μ-calculus as bisimulations do. Similarly to Example 2.3.4, let $\emptyset$-interpretation m correspond to K and n correspond to K'. The relation $\rho \subseteq S \times S'$ gives rise to an abstraction γ from m to n such that $\gamma_*(X) = \{s' \in S' \mid \forall s \in S.(s, s') \in \rho \Rightarrow s \in X\}$.

Example 2.5.6 (Galois connection)**.** Let $\mathbf{Atm}$ be $\{\Diamond\}$. Similarly to Example 2.3.3, a transition system $(\wp(\mathbf{Z}), R)$ gives rise to the $\emptyset$-interpretation $n \colon \Sigma_{\mathbf{Atm}} \to U\mathbf{Pos}_{\mathbf{CL}}$. There exists the following Galois connection (α_*, γ_*) between $\wp(\mathbf{Z})$ and $\mathbf{Intvl}$ of Example 2.3.6.

- $\alpha_*(X) = \vee_{\mathbf{Intvl}}\{(z, z) \mid z \in X\}$
- $\gamma_*(i) = \{z \in \mathbf{Z} \mid (z, z) \leq_{\mathbf{Intvl}} i\}$

A monotone function $i \mapsto \vee_{\mathbf{Intvl}}\{(z, z) \mid z \in \mathbf{Z}, \exists z' \in \mathbf{Z}. zRz' \wedge (z', z') \leq_{\mathbf{Intvl}} i\}$ gives rise to the $\emptyset$-interpretation $m \colon \Sigma_{\mathbf{Atm}} \to U\mathbf{Pos}_{\mathbf{CL}}$. The Galois connection gives rise to an abstraction γ from m to n.

Corollary 2.5.7 (Formula-preservation)**.** For abstraction $\gamma \colon m \to n$ and $R\mu$-formula φ, $[\![\varphi]\!]_n \circ \bar{\gamma}_* \leq \bar{\gamma}_* \circ [\![\varphi]\!]_m$.

This corollary implies the formula-preservation theorem in two senses. Let α_* be the left adjoint of γ_*. On the one hand, this corollary implies $\alpha_* \circ [\![\varphi]\!]_n \circ \gamma_* \leq [\![\varphi]\!]_m$. Then, $[\![\varphi]\!]_m$ is a top element, if $[\![\varphi]\!]_n$ is so and α_* preserves the top element. In this sense, n is considered as an abstract interpretation to m. On the other hand, this corollary also implies $[\![\varphi]\!]_n \leq \gamma_* \circ [\![\varphi]\!]_m \circ \alpha_*$. Then, $[\![\varphi]\!]_n$ is a bottom element, if $[\![\varphi]\!]_m$ is so and γ_* preserves the bottom element. In this sense, m is considered as an abstract interpretation to n. Since **RMu**-algebras do not always have the negation operator, properties for the top element are independent from properties for the bottom element. Therefore, the difference between these two senses is meaningful.

The next theorem gives a construction of an abstract $\emptyset$-interpretation from a concrete interpretation. This is a generalisation of the typical construction when a program is model checked using abstract-data mapping [8], predicate abstraction [13], or abstract interpretation [9].

Theorem 2.5.8 (Construction of abstract interpretation)**.** Let M be an **RMu**-algebra and m be a $\emptyset$-interpretation $m\colon \Sigma_{\mathbf{Atm}} \to UM$. For object $n*$ of UM and right adjoint arrow $\gamma_*\colon\ m* \to n*$, the data $n*$ extends to an $\emptyset$-interpretation $n\colon\ \Sigma_{\mathbf{Atm}} \to UM$ that makes γ_* an abstraction $\gamma\colon\ m \to n$.

Proof. Let $\alpha_*\colon\ n* \to m*$ be a left adjoint of γ_*. The arrow part of n is given by the function which sends $a \in \mathbf{Atm}$ to $\gamma_* \circ ma \circ \alpha_*$. The adjointness make γ lax natural. □

Our definition of $\emptyset$-interpretations requires neither the negations nor duality of modal operators. That is why we can construct abstract interpretations by Theorem 2.5.8. Note that an abstract interpretation does not have to have the similar structure to the concrete interpretation.

Example 2.5.9 (From Kripke structure to another structure)**.** Let **Atm** be $P \cup \{\neg p \mid p \in P\} \cup \{\Diamond, \Box\}$. Similarly to Example 2.3.5, let $\emptyset$-interpretation m correspond to a Kripke model $K = (S, R, Q)$. For set B and a relation $\rho \subseteq S \times B$, the inverse function $\rho^{-1}\colon\ \wp(B) \to \wp(S)$ has a right adjoint $\gamma_*(X) = \{b \in B \mid \forall s \in S.(s, b) \in \rho \Rightarrow s \in X\}$. By Theorem 2.5.8, we get the following $\emptyset$-interpretation n.

$$n* = \wp(B)$$

$np\colon X \mapsto \{b \in B \mid \forall s \in S.(s,b) \in \rho \Rightarrow p \in Q(s)\}$ for $p \in P$

$n\neg p\colon X \mapsto \{b \in B \mid \forall s \in S.(s,b) \in \rho \Rightarrow p \notin Q(s)\}$ for $p \in P$

$n\Diamond\colon X \mapsto \{b \in B \mid \forall s \in S.(s,b) \in \rho \Rightarrow \exists s' \in S.\exists x \in X.(s,s') \in R, (s',x) \in \rho\}$

$n\Box\colon X \mapsto \{b \in B \mid \forall s \in S.(s,b) \in \rho \Rightarrow \forall s' \in S.((s,s') \in R \Rightarrow \exists x \in X.(s',x) \in \rho)\}$

Here, $n\neg p$ is not always a complement of np, although $m\neg p$ is a complement of mp. (When $(s,b) \in \rho$, $(s',b) \in \rho$, $p \in Q(s)$, and $p \notin Q(s)$) Moreover, $n\Box$ does not always preserve intersections, although $m\Box$ preserves intersections. (When $S = \{*\}$, $R = \{(*,*)\}$, $B = \{1,2\}$, and $\rho = \{(*,1),(*,2)\}$)

2.6 Comparison with Modal μ-Calculus

In this section, we compare our logic $R\mu$ with modal μ-calculus $L\mu$ [26] and CTL [6]. First, we introduce the syntactic restriction $L\mu^-$ of $L\mu$. Next, we prove that $L\mu^-$ can be translated in $R\mu$ and that CTL can be translated in $L\mu^-$.

Definition 2.6.1. Let $P_{L\mu^-}$ be a set of propositional constants and let **Var** be a set of propositional variables. We write $FV(\varphi)$ for the set of free propositional variables in φ. $L\mu^-$-*formulas* are given by the grammar

$$
\begin{array}{lllll}
\varphi & ::= & p & \mid \neg p & (p \in P_{L\mu^-}) \\
& \mid & \Diamond\varphi & \mid \Box\varphi & \\
& \mid & Z & & (Z \in \mathbf{Var}) \\
& \mid & \bot & \mid \top & \\
& \mid & \varphi \vee \varphi & \mid \varphi \wedge \varphi & \\
& \mid & \mu Z.(\varphi \vee \varphi) & \mid \nu Z.(\varphi \wedge \varphi) &
\end{array}
$$

where

- $\varphi_1 \vee \varphi_2$ and $\varphi_1 \wedge \varphi_2$ must satisfy that $FV(\varphi_1) = \emptyset$, $FV(\varphi_2) = \emptyset$, or $FV(\varphi_1) = FV(\varphi_2)$.

- $\mu Z.(\varphi_1 \vee \varphi_2)$ and $\nu Z.(\varphi_1 \wedge \varphi_2)$ must satisfy that $Z \notin FV(\varphi_1)$ and that $FV(\varphi_2) \subseteq \{Z\}$.

For $L\mu^-$-formulas φ and ψ, the result $\psi[\varphi/Z]$ of capture-avoiding substitution of φ for Z in ψ is an $L\mu^-$-formula. An arbitrary $L\mu^-$-formula φ has at most one free variable. Two free variables can appear in the part $\varphi_1 \vee \varphi_2$ of $\mu Z.(\varphi_1 \vee \varphi_2)$. However, the part $\varphi_1 \vee \varphi_2$ is not an $L\mu^-$-formula. Since three free variables can not appear in the part $\varphi_1 \vee \varphi_2$, an $L\mu^-$-formula can be α-converted to a formula with at most two variables. For example, the followings are $L\mu^-$-formulas.

$$\mu v_0.(\nu v_1.(p \wedge \mu v_0.(v_1 \vee \Diamond v_0)) \vee \Diamond v_0)$$

$$\mu v_0.(p \vee \Diamond \nu v_1.(v_0 \wedge \Box \mu v_2.(v_1 \vee \Diamond \nu v_3.(v_2 \wedge \Box v_3))))$$

The following $L\mu$-formula is not an $L\mu^-$-formula.

$$\nu v_1.(p \wedge \mu v_0.(v_1 \wedge \Diamond v_0))$$

We define two translations between $L\mu^-$-formulas and $R\mu$-formulas. Our translations assume the following signature of $R\mu$.

$$\mathbf{Atm} \quad = \quad P_{L\mu^-} \cup \{\neg p \mid p \in P_{L\mu^-}\} \cup \{\Box, \Diamond\}$$

Under the signature, the meaning of the propositional constants is specified by the theory $\Delta_{\mathbf{const}}$, which consists of the following axioms. For $R\mu$-formula φ, $p \circ \varphi = p$ is a $\Delta_{\mathbf{const}}$-theorem.

$$\begin{array}{ll} p \circ \bot = p & (p \in P_{L\mu^-}) \\ \neg p \circ \bot = \neg p & (p \in P_{L\mu^-}) \end{array}$$

This is the translation from an $L\mu^-$-formula to an $R\mu$-formula. It preserves the α-equality of $L\mu^-$-formulas.

Definition 2.6.2 ($L\mu^-$-formula to $R\mu$-formula)**.** For $L\mu^-$-formula φ, $R\mu$-formula

$|\varphi|$ is given by the following

$$
\begin{array}{rclrcl}
|\bot| & = & \bot & |p| & = & p \\
|\top| & = & \top & |\neg p| & = & \neg p \\
|\varphi \vee \psi| & = & |\varphi| \vee |\psi| & |\Diamond\varphi| & = & \Diamond \circ |\varphi| \\
|\varphi \wedge \psi| & = & |\varphi| \wedge |\psi| & |\Box\varphi| & = & \Box \circ |\varphi| \\
|\mu Z.(\varphi \vee \psi)| & = & \mu(|\varphi|, |\psi|) & |Z| & = & \mathbf{Id} \\
|\nu Z.(\varphi \wedge \psi)| & = & \nu(|\varphi|, |\psi|) & & &
\end{array}
$$

Since each structures of $R\mu$ are preserved by precomposition, this lemma holds.

Lemma 2.6.3. If $FV(\psi) \subseteq \{Z\}$, then $|\psi[\varphi/Z]| = |\psi| \circ |\varphi|$ is a $\Delta_{\mathbf{const}}$-theorem.

Proof. By induction on the structure of ψ, we can prove that. □

The translation is faithful with respect to the Kripke semantics in the following sense. The Kripke semantics of $L\mu^-$ is the same as that for $L\mu$; we write $K, s \models_{L\mu^-} \varphi$ when a state s satisfies φ in a Kripke structure K. Given a Kripke structure K, we define the $\emptyset$-interpretation $m_K\colon \Sigma_{\mathbf{Atm}} \to U\mathbf{Pos}_{\mathbf{CL}}$ by Example 2.3.5.

Theorem 2.6.4. For closed $L\mu^-$-formulas φ,

$$[\![|\varphi|]\!]_{m_K}(X) = \{s \mid K, s \models_{L\mu^-} \varphi\}$$

Conversely, $R\mu$-formulas are translated into $L\mu^-$-formulas.

Definition 2.6.5 ($R\mu$-formula to $L\mu^-$-formula)**.** Let i be 0 or 1. For i and $R\mu$-formula φ, $L\mu^-$-formula $|\varphi|^{-1}_{v_i}$ such that $FV(|\varphi|^{-1}_{v_i}) \subseteq \{v_i\}$ is given by the following

$$
\begin{array}{rclrcl}
|\bot|^{-1}_{v_i} & = & \bot & |p|^{-1}_{v_i} & = & p \\
|\top|^{-1}_{v_i} & = & \top & |\neg p|^{-1}_{v_i} & = & \neg p \\
|\varphi \vee \psi|^{-1}_{v_i} & = & |\varphi|^{-1}_{v_i} \vee |\psi|^{-1}_{v_i} & |\Diamond|^{-1}_{v_i} & = & \Diamond v_i \\
|\varphi \wedge \psi|^{-1}_{v_i} & = & |\varphi|^{-1}_{v_i} \wedge |\psi|^{-1}_{v_i} & |\Box|^{-1}_{v_i} & = & \Box v_i \\
|\mu(\varphi, \psi)|^{-1}_{v_i} & = & \mu v_{1-i}.(|\varphi|^{-1}_{v_i} \vee |\psi|^{-1}_{v_{1-i}}) & |\mathbf{Id}|^{-1}_{v_i} & = & v_i \\
|\nu(\varphi, \psi)|^{-1}_{v_i} & = & \nu v_{1-i}.(|\varphi|^{-1}_{v_i} \wedge |\psi|^{-1}_{v_{1-i}}) & |\varphi \circ \psi|^{-1}_{v_i} & = & |\varphi|^{-1}_{v_0}[|\psi|^{-1}_{v_i}/v_0]
\end{array}
$$

These translations give equivalence between $R\mu$-formulas and $L\mu^-$-formulas as follows.

Theorem 2.6.6. For i and $R\mu$-formula φ, $||\varphi|_{v_i}^{-1}| = \varphi$ is a $\Delta_{\mathbf{const}}$-theorem. For i and $L\mu^-$-formula φ satisfying $FV(\varphi) \subseteq \{v_i\}$, $||\varphi||_{v_i}^{-1}$ is α-equal to φ.

Proof. By Lemma 2.6.3 and induction on φ, we can prove the two propositions. $\square$

Lastly, we compare CTL-formula with $L\mu^-$-formula. The semantics of CTL is given by Kripke structures with total transition relations. The translation $|| - ||$ from CTL-formulas in the negation normal form to closed modal μ-formula is well-known [6]. It is direct to check that $||\varphi||$ is an $L\mu^-$-formula for any negation normal CTL-formula φ.

$$\begin{aligned}
||p|| &= p \\
||\neg p|| &= \neg p \\
||\mathbf{EX}\varphi|| &= \Diamond||\varphi|| \\
||\mathbf{AX}\varphi|| &= \Box||\varphi|| \\
||\mathbf{EF}\varphi|| &= \mu Z.(||\varphi|| \vee \Diamond Z) \\
||\mathbf{AF}\varphi|| &= \mu Z.(||\varphi|| \vee \Box Z) \\
||\mathbf{E}(\varphi\mathbf{U}\psi)|| &= \mu Z.(||\psi|| \vee (||\varphi|| \wedge \Diamond Z)) \\
||\mathbf{A}(\varphi\mathbf{U}\psi)|| &= \mu Z.(||\psi|| \vee (||\varphi|| \wedge \Box Z)) \\
||\mathbf{EG}\varphi|| &= \nu Z.(||\varphi|| \wedge \Diamond Z) \\
||\mathbf{AG}\varphi|| &= \nu Z.(||\varphi|| \wedge \Box Z) \\
||\mathbf{E}(\varphi\mathbf{V}\psi)|| &= \nu Z.(||\psi|| \wedge (||\varphi|| \vee \Diamond Z)) \\
||\mathbf{A}(\varphi\mathbf{V}\psi)|| &= \nu Z.(||\psi|| \wedge (||\varphi|| \vee \Box Z))
\end{aligned}$$

2.7 Related Algebraic Structures

Various algebraic structures for fixed point logics have been developed. However, these algebraic structures do not always correspond to Lawvere A-theories.

There exists the Lawvere **Set**-theory for which the models are all Boolean algebras. A complete Boolean algebra with a Galois connection on it is called *Galois algebras* [50]. LTL and CTL are axiomatised by using them. However, there exists no Lawvere **Set**-theory for which the models are all complete Boolean algebras [17].

A Boolean algebra with a meet-preserving function on it is called *modal algebra* [45]. There exists the Lawvere **Set**-theory for which the models are all modal algebras. A *modal μ-algebra* [2] is a modal algebra which has least fixed points of certain functions defined by using formulas and assignments of modal μ-calculus [26]. The algebraic structure of modal μ-algebras is a complete axiomatisation of modal μ-calculus. However, it is not equational. Although Santocanale proved that modal μ-calculus has a complete equational axiomatisation [46], he introduce the notion of polarised signature and fixed point terms [1] in order to define a signature in usual sense. The polarised signature is not finite. In fact, he also proved that the theory for modal μ-calculus is not equationally finitely based [46].

Kleene algebra [28, 27] has least fixed points of certain monotone functions. However, it is axiomatised by not purely equational theory, but equational Horn theory. *Action algebra* [43] can be axiomatised by purely equational theory. Therefore, there exists the Lawvere **Set**-theory for which the models are all action algebras. There exists a forgetful functor from the category of Kleene algebras to the category of action algebras, which has a left adjoint [12].

Kleene category has been proposed by Y. Kinoshita [24] in 2001 and independently reintroduced by W. Kahl [18] in 2004. Although the structure of Kleene category is symmetric to the composition, it is not symmetric to the order. When the structure of top elements 1 is added to it similarly to bottom elements 0, they satisfy $1 = 1 \circ 0 = 0$. It contains no interesting examples.

On the other hand, although the structure of **RMu**-algebra is not symmetric to the composition, it is symmetric to the order. The precise relationship between them is given by using the following structures.

Theorem 2.7.1. The set of Kleene categories is isomorphic to the set of locally ordered categories such that

1. for objects x and y, the hom-poset is a bounded semilattice

$$(C(x, y), \leq_{x,y}, \vee_{x,y}, \perp_{x,y});$$

2. for objects x, y, z, and arrow $f\colon x \to y$, the pre-composition function $C(f, z)\colon C(y, z) \to C(x, z)$ preserves the structure of the bounded semilattice;

3. for arrows $f\colon x \to y$ and $g\colon y \to y$, $C(x, y)$ has a least fixed point $\mu_{x,y}(f, g)$ of the monotone function which sends $\sigma\colon x \to y$ to $f \vee_{x,y} (g \circ \sigma)\colon x \to y$;

4. for arrows $h\colon x \to y$, $f\colon y \to z$, and $g\colon z \to z$, $\mu_{y,z}(f, g) \circ h = \mu_{x,z}(f \circ h, g)$;

5. for objects x, y, z, and arrow $f\colon y \to z$, the post-composition function $C(x, f)\colon C(x, y) \to C(x, z)$ preserves the structure of the bounded semilattice; and

6. for arrows $f\colon x \to y$ and $g\colon x \to x$, $f \circ \mu_{x,x}(\mathbf{Id}, g)$ is a least fixed point of the monotone function which sends $\sigma\colon x \to y$ to $f \vee_{x,y} (\sigma \circ g)\colon x \to y$.

Proof. (sketch) A locally ordered category satisfying the above conditions can define g^* by $\mu_{x,x}(\mathbf{Id}, g)$. Conversely, a Kleene category can define $\mu_{x,y}(f, g)$ by $g^* \circ f$. □

An arbitrary **RMu**-algebra satisfies the conditions 1 through 4. $\mathbf{Pos_{CL}}$ is not a Kleene category. Consider the following transition system (S, R).

$$\begin{aligned} S &= \{1, 2, 3, 4, 5\} \\ R &= \{(1, 2), (2, 3), (3, 3), (1, 4), (4, 5), (5, 5)\} \end{aligned}$$

Let $\Box$ be the function $\Box\colon \wp(S) \to \wp(S)$ which sends X to $\{s \in S \mid \forall s' \in S.(s, s') \in R \Rightarrow s' \in X\}$. Since $\Box$ is a monotone, it is an arrow in $\mathbf{Pos_{CL}}$. We show a counterexample of the sixth condition as follows. Put $x = y = \wp(S)$, $f = \mathbf{Id}$, $g = \Box$, and $\sigma = \mathbf{Id} \vee_{\wp(S),\wp(S)} \Box \vee_{\wp(S),\wp(S)} (\Box \circ \Box)$. They satisfy $f \leq_{\wp(S),\wp(S)} \sigma$ and $\sigma \circ g \leq_{\wp(S),\wp(S)} \sigma$. However, $1 \in \mu_{\wp(S),\wp(S)}(\mathbf{Id}, g)(\{2, 5\})$ and $1 \notin \sigma(\{2, 5\})$. Therefore, they do not satisfy $f \circ \mu_{\wp(S),\wp(S)}(\mathbf{Id}, g) \leq_{\wp(S),\wp(S)} \sigma$.

By the same way as Section 4.5.1, one should give Lawvere **LocOrd**-theory for which the models are all Kleene categories.

Chapter 3

Construction of Complex Interpretations and Abstractions

In this chapter, we formulate a construction method of a complex interpretation of $R\mu$ from a simpler one in algebraic approach.

The leading example is the following program.

```
/* 1 */
while(0 =< x){
    /* 2 */
    x = x+y;
    /* 3 */
}
/* 4 */
```

Here x and y are integer variables. We aim to show that the line 4 in the program is not reachable if x and y are positive in the initial line 1. Although this example is simple enough to be verified directly, we use it to explain our compositional verification method that applies to larger problems. The semantics of the program as an interpretation of $R\mu$, we take the following signature **Atm** and the theory Δ.

$$\begin{aligned} \mathbf{Atm} &= \{\mathbf{isn't1},\ \mathbf{isn't4},\ (\mathbf{x} < \mathbf{0}),\ (\mathbf{y} < \mathbf{0}),\ \Box\} \\ \Delta &= \emptyset \end{aligned}$$

The property to be verified can be expressed as the following $R\mu$-formula in the signature **Atm**.

$$\mathbf{isn't1} \vee (\mathbf{x} < \mathbf{0}) \vee (\mathbf{y} < \mathbf{0}) \vee \nu(\mathbf{isn't4}, \Box)$$

The concrete interpretation of $\Box$ is based on the state set $\{1,2,3,4\} \times \mathbf{Z} \times \mathbf{Z}$ and a transition relation R on the state set. However, it is complicated to give R directly.

$$\begin{aligned} R \quad = \quad & \{((1,a,b),(2,a,b)) \mid 0 \leq a\} \cup \\ & \{((1,a,b),(4,a,b)) \mid a < 0\} \cup \\ & \{((2,a,b),(3,a',b)) \mid a + b = a'\} \cup \\ & \{((3,a,b),(2,a,b)) \mid 0 \leq a\} \cup \\ & \{((3,a,b),(4,a,b)) \mid a < 0\} \end{aligned}$$

This chapter gives an algebraic structure to construct an interpretation of programs from that of primitive commands. We formulate programs as *GLTS(generalised labelled transition system)-formulas* and formulate primitive commands as *labels*.

Transition systems are usually modelled in terms of relations over some sets. We regard a fibration as a general structure to organise them. Accordingly, we define a fibration with a certain additional operations that interpret GLTS-formula construction. We call such fibrations **GLTS**-*algebras*. An interpretation of primitive commands, or labels, is given by a functor into a **GLTS**-algebra. This extends to the interpretation of all the GLTS-formulas by the theorem showing that GLTS-formulas are objects of a free **GLTS**-algebra, similarly to the case of $R\mu$.

We formulate abstract interpretations of labels. An abstract interpretation of labels extends to an abstract interpretation of a GLTS-formula. To naturally extend it, this chapter also use 2-categorical formulation of abstraction. The leading example of abstraction is based on the following function.

$$\begin{aligned} \mathbf{pos} \quad &: \quad \mathbf{Z} \to \mathbf{Bool} = \{\mathbf{t}, \mathbf{f}\} \\ \mathbf{pos}(x) \quad &= \quad \begin{cases} \mathbf{t} & (0 \leq x) \\ \mathbf{f} & (x < 0) \end{cases} \end{aligned}$$

3.1 GLTS-formulas

We give the syntax of GLTS-formulas.

The syntax of GLTS-formulas is parametrised by *signature* that consists of three sets $P_{\mathbf{Pc}}$, $P_{\mathbf{Test}}$, and $P_{\mathbf{Com}}$. Elements of $P_{\mathbf{Pc}}$ are predicate symbols on the program counter, $P_{\mathbf{Test}}$ test expressions, and $P_{\mathbf{Com}}$ primitive commands, respectively. *GLTS-formulas* are given by the following grammar.

$$
\begin{array}{rcll}
R & ::= & \mathbf{From}(p) & (p \in P_{\mathbf{Pc}}) \\
  & | & \mathbf{To}(p) & (p \in P_{\mathbf{Pc}}) \\
  & | & \mathbf{If}(t) & (t \in P_{\mathbf{Test}}) \\
  & | & \mathbf{Do}(c) & (c \in P_{\mathbf{Com}}) \\
  & | & \bot & \\
  & | & \top & \\
  & | & R \vee R & \\
  & | & R \wedge R &
\end{array}
$$

A GLTS-formula is similar to a propositional formula, except for the lack of negation. One can consider a GLTS-formula as the propositional formula expressing a transition relation. For example, this program

```
/* 1 */
while(0 =< x){
    /* 2 */
    x = x+y;
    /* 3 */
}
/* 4 */
```

is represented by the following GLTS-formula **Loopxy** with signatures.

$$
\begin{array}{rcl}
P_{\mathbf{Pc}} & = & \{\mathbf{is1}, \mathbf{is2}, \mathbf{is3}, \mathbf{is4}\} \\
P_{\mathbf{Test}} & = & \{\mathbf{x >= 0}, \mathbf{x < 0}, \mathbf{y >= 0}\} \\
P_{\mathbf{Com}} & = & \{\mathbf{x = x + y}, \mathbf{skip}\}
\end{array}
$$

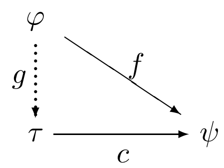
$$\begin{aligned}
\mathbf{Loopxy} \quad = \quad & (\mathbf{From}(\mathbf{is1}) \wedge \mathbf{To}(\mathbf{is2}) \wedge \mathbf{If}(\mathbf{x} >= \mathbf{0}) \wedge \mathbf{Do}(\mathbf{skip}))\vee \\
& (\mathbf{From}(\mathbf{is1}) \wedge \mathbf{To}(\mathbf{is4}) \wedge \mathbf{If}(\mathbf{x} < \mathbf{0}) \wedge \mathbf{Do}(\mathbf{skip}))\vee \\
& (\mathbf{From}(\mathbf{is2}) \wedge \mathbf{To}(\mathbf{is3}) \wedge \mathbf{Do}(\mathbf{x} := \mathbf{x} + \mathbf{y}))\vee \\
& (\mathbf{From}(\mathbf{is3}) \wedge \mathbf{To}(\mathbf{is2}) \wedge \mathbf{If}(\mathbf{x} >= \mathbf{0}) \wedge \mathbf{Do}(\mathbf{skip}))\vee \\
& (\mathbf{From}(\mathbf{is3}) \wedge \mathbf{To}(\mathbf{is4}) \wedge \mathbf{If}(\mathbf{x} < \mathbf{0}) \wedge \mathbf{Do}(\mathbf{skip}))
\end{aligned}$$

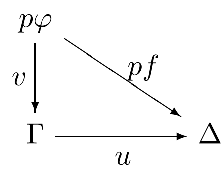
Section 3.3 gives the intended interpretation of **Loopxy**.

3.2 Cloven Fibration

We give interpretations of GLTS-formula by using algebraic approach. Transition systems are usually modelled in terms of relations over some sets. We regard a fibration [16, 14] as a general structure to organise them.

Let E and B be arbitrary categories and let $p\colon E \to B$ be a functor. An arrow $c\colon \tau \to \psi$ in E is *cartesian* over $u = pc\colon \Gamma \to \Delta$ in B if, for $f\colon \varphi \to \psi$ in E and $v\colon p\varphi \to \Gamma$ in B satisfying $pf = u \circ v$, there exists a unique arrow $g\colon \varphi \to \tau$ in E satisfying $pg = v$ and $f = c \circ g$.





A functor $p\colon E \to B$ is a *fibration* if, for ψ in E and $u\colon \Gamma \to p\psi$ in B, there exists an object τ in E and a cartesian arrow $c\colon \tau \to \psi$ over u. We call E the *total category* and B the *base category* of a fibration $p\colon E \to B$. For fibration $p\colon E \to B$, a *cleavage* for p is a particular choice of cartesian arrow $\overline{u}(\psi)\colon u^*\psi \to \psi$ for each ψ and u. A fibration equipped with a particular cleavage is called a *cloven fibration*.

Here, we show a leading example of fibrations. Let $\mathbf{Sub}(\mathbf{Set})$ be the category given as follows.

- An object is a pair (X, I) of objects $X, I \in \mathbf{Set}$ satisfying $X \subseteq I$.
- An arrow from (X, I) to (Y, J) is an arrow $f \colon I \to J \in \mathbf{Set}$ such that $f(x) \in Y$ for every $x \in X$.

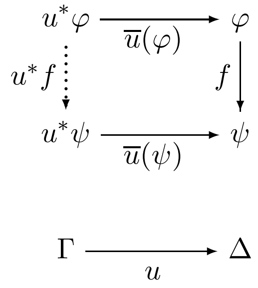
Let $\mathbf{base} \colon \mathbf{Sub}(\mathbf{Set}) \to \mathbf{Set}$ be the functor which sends (X, I) to I. We show that $\mathbf{base}$ is a cloven fibration. For $(Y, J) \in \mathbf{Sub}(\mathbf{Set})$ and $u \colon I \to J$, the set $u^{-1}(Y) = \{x \in I \mid u(x) \in Y\}$ is a subset of I. Let $u^*(Y, J)$ be $(u^{-1}(Y), I)$ and $c \colon u^*(Y, J) \to (Y, J)$ be the function that sends $x \in I$ to $u(x) \in J$. It is trivial that $pc = u$. Let $f \colon (Z, K) \to (Y, J)$ satisfy $u \circ v = \mathbf{base} f$ for some $v \colon K \to I$. Let z be an element of Z. By the condition of f, $f(z)$ is an element of Y. By the equation $u \circ v = \mathbf{base} f$, $u(v(z))$ is also an element of Y. By the definition of $u^{-1}(Y)$, $v(z)$ is an element of $u^{-1}(Y)$. So, we can take $g \colon (Z, K) \to (u^{-1}(Y), I)$ sending z to $v(z)$. It is trivial that $pg = v$ and $c \circ g = f$. By the condition $pg = v$, the uniqueness is trivial. Therefore, $\mathbf{base}$ is a cloven fibration. We call it the *subset fibration*.

For cloven fibration $p \colon E \to B$ and each object Γ in B, the *fibre* E_Γ over Γ is the following category.

- object: object ψ in E such that $p\psi = \Gamma$
- arrow: arrow f in E such that $pf = \mathbf{Id}_\Gamma$

An arrow $u \colon \Gamma \to \Delta$ in B determines a *reindexing functor* $u^* \colon E_\Delta \to E_\Gamma$ as follows.

- on objects, ψ in E_Δ is sent to the domain $u^*\psi$ of the cartesian arrow $\overline{u}(\psi) \colon u^*\psi \to \psi$.
- on arrows, for $f \colon \varphi \to \psi$ in E_Δ, $u^* f$ is the unique arrow such that $p(u^* f) = \mathbf{Id}$ and $\overline{u}(\psi) \circ u^* f = f \circ \overline{u}(\varphi)$.



For $I \in \mathbf{Set}$, the fibre $\mathbf{Sub}(\mathbf{Set})_I$ of the subset fibration is the partially ordered set $(\wp(I), \subseteq)$. The reindexing functor u^* of $u\colon I \to J$ is the monotone function that sends $Y \in \wp(J)$ to $u^{-1}(Y) \in \wp(I)$.

A fibration $p\colon E \to B$ has a *fibred terminal object* if each fibre E_Γ has a terminal object and each reindexing functor preserves them. The subset fibration **base** has fibred terminal objects, since the fibre $\mathbf{Sub}(\mathbf{Set})_I$ has a terminal object (I, I) and $u\colon I \to J$ satisfies $u^{-1}(J) = I$.

A fibration $p\colon E \to B$ has *fibred finite limits* if each fibre E_Γ has finite limits preserved by each reindexing functor. The subset fibration **base** has fibred pullbacks, since the fibre $\mathbf{Sub}(\mathbf{Set})_I$ has a pullback $(X_1, I) \times_{f,g} (X_2, I) = (X_1 \cap X_2, I)$ and $u\colon I \to J$ satisfies $u^{-1}(Y_1 \cap Y_2) = u^{-1}(Y_1) \cap u^{-1}(Y_2)$. Since the subset fibration has a fibred terminal object and fibred pullbacks, it has fibred finite limits.

A fibration $p\colon E \to B$ has *fibred finite colimits* if each fibre E_Γ has finite colimits preserved by each reindexing functor. The subset fibration **base** has fibred finite colimits, since the fibred initial object is the empty set and a fibred pushout is given by an union of sets.

Let B be a category with pullbacks. A cloven fibration $p\colon E \to B$ is *cocomplete* [16] if

1. p has fibred finite colimits;

2. for arrow $u\colon \Gamma \to \Delta$ in B, the induced reindexing functor u^* has a left adjoint $\Sigma_u\colon E_\Gamma \to E_\Delta$;

3. the Beck-Chevalley condition holds: for pullback square

$$\begin{array}{ccc} \Lambda & \xrightarrow{\;y\;} & \Gamma \\ {\scriptstyle w}\downarrow & \lrcorner & \downarrow{\scriptstyle u} \\ \Xi & \xrightarrow[\;x\;]{} & \Delta \end{array}$$

   in B, the canonical natural transformation $\Sigma_w y^* \Rightarrow x^* \Sigma_u\colon E_\Gamma \to E_\Xi$ is an isomorphism.

We show that the subset fibration $\mathbf{base}\colon \mathbf{Sub}(\mathbf{Set}) \to \mathbf{Set}$ is cocomplete. Sets $X \subseteq I$ and $Y \subseteq J$ satisfy $X \subseteq u^{-1}(Y)$ if and only if they satisfy $\{u(i) \mid i \in X\} \subseteq Y$.

Therefore, u^* has the left adjoint Σ_u that sends X to $\{u(i) \mid i \in X\}$. The base category **Set** has pullbacks. The pullback of $f\colon K \to J$ and $u\colon I \to J$ is the set $\{(k,i) \in K \times I \mid f(k) = u(i)\}$. The functions f^*u and u^*f are projections.

$$\begin{array}{ccc} K \times_{f,u} I & \xrightarrow{u^*f} & I \\ {\scriptstyle f^*u}\downarrow & \lrcorner & \downarrow{\scriptstyle u} \\ K & \xrightarrow[f]{} & J \end{array}$$

The Beck-Chevalley condition is equivalent to that $f^{-1}(\Sigma_u(X))$ is equal to $\Sigma_{f^*u}((u^*f)^{-1}(X))$ for every $X \subseteq I$. The condition is satisfied as follows.

$$\begin{array}{rcl} f^{-1}(\Sigma_u(X)) & = & f^{-1}(\{u(i) \mid i \in X\}) \\ & = & \{k \in K \mid \exists i \in X.u(i) = f(k)\} \\ & = & \Sigma_{f^*u}(\{(k,i) \in K \times I \mid f(k) = u(i), i \in X\}) \\ & = & \Sigma_{f^*u}((u^*f)^{-1}(X)) \end{array}$$

Let B be a category with pullbacks. A fibration $p\colon E \to B$ is *complete* if

1. p has fibred finite limits;
2. for arrow $u\colon \Gamma \to \Delta$ in B, the induced reindexing functor u^* has a right adjoint $\Pi_u\colon E_\Gamma \to E_\Delta$;
3. the Beck-Chevalley condition holds: for pullback square

$$\begin{array}{ccc} \Lambda & \xrightarrow{y} & \Gamma \\ {\scriptstyle w}\downarrow & \lrcorner & \downarrow{\scriptstyle u} \\ \Xi & \xrightarrow[x]{} & \Delta \end{array}$$

   in B, the canonical natural transformation $x^*\Pi_u \Rightarrow \Pi_w y^*\colon E_\Gamma \to E_\Xi$ is an isomorphism.

The subset fibration **base**: **Sub**(**Set**) $\to$ **Set** is complete. For $u\colon I \to J$, $\Pi_u(X)$ is defined to be $\{j \in J \mid \forall i \in I.u(i) = j \Rightarrow i \in X\}$. We prove that Π_u is the right adjoint of u^*, as follows.

$$\begin{array}{rcl} u^{-1}(Y) \subseteq X & \Leftrightarrow & \forall i \in I. u(i) \in Y \Rightarrow i \in X \\ & \Leftrightarrow & \forall i \in I. \forall j \in Y. (u(i) = j \Rightarrow i \in X) \\ & \Leftrightarrow & \forall j \in Y. \forall i \in I. (u(i) = j \Rightarrow i \in X) \\ & \Leftrightarrow & Y \subseteq \Pi_u(X) \end{array}$$

The Beck-Chevalley condition is equivalent to that $f^{-1}(\Pi_u(X))$ is equal to $\Pi_{f^*u}((u^*f)^{-1}(X))$ for every $X \subseteq I$. The condition is satisfied as follows.

$$\begin{array}{rcl} f^{-1}(\Pi_u(X)) & = & f^{-1}(\{j \in J \mid \forall i \in I. u(i) = j \Rightarrow i \in X\}) \\ & = & \{k \in K \mid \forall i \in I. u(i) = f(k) \Rightarrow i \in X\} \\ & = & \{k \in K \mid \forall i \in I. u(i) = f(k) \Rightarrow (k, i) \in (u^*f)^{-1}(X)\} \\ & = & \Pi_{f^*u}((u^*f)^{-1}(X)) \end{array}$$

A fibration $p\colon E \to B$ is a *fibred CCC* if each fibre E_Γ is a cartesian closed category and each reindexing functor preserves finite products and exponentials. The subset fibration **base** is a fibred CCCs. The fibre $\mathbf{Sub}(\mathbf{Set})_I$ has an exponential $[(X_1, I), (X_2, I)] = (\{i \in I \mid i \notin X_1\} \cup X_2, I)$ and $u\colon I \to J$ satisfies the following equation.

$$\begin{array}{rcl} u^*([(Y_1, J), (Y_2, J)]) & = & (u^{-1}(\{j \in J \mid j \notin Y_1\} \cup Y_2), I) \\ & = & (u^{-1}(\{j \in J \mid j \notin Y_1\}) \cup u^{-1}(Y_2), I) \\ & = & (\{i \in I \mid u(i) \notin Y_1\} \cup u^{-1}(Y_2), I) \\ & = & (\{i \in I \mid i \notin u^{-1}(Y_1)\} \cup u^{-1}(Y_2), I) \\ & = & [u^*(Y_1, J), u^*(Y_2, J)] \end{array}$$

3.3 Algebraic Semantics of GLTS-formula

To give an interpretation of GLTS-formulas, we now define the structure we call **GLTS**-algebras on cloven fibrations.

Definition 3.3.1. A **GLTS**-*algebra* is a cloven fibration that is cocomplete and has

- finite colimits in the base category,

- finite limits in the base category, and
- fibred finite limits.

A **GLTS**-algebra has the structure to give interpretations of GLTS-formulas. The leading example is the subset fibration **base**.

Section 4.6 shows that **GLTS**-algebras are models of a Lawvere A-theory, **GLTS**, with A being the following category $\mathbf{Cat}_o^{\rightarrow}$.

- An object of $\mathbf{Cat}_o^{\rightarrow}$ consists of two small categories with a functor between them.
- For objects $q\colon D \to A$ and $p\colon E \to B$, an arrow $G\colon q \to p$ consists of a functor $G_0\colon D \to E$ and a functor $G_1\colon A \to B$ such that $p \circ G_0 = G_1 \circ q$.

$$\begin{array}{ccc} D & \xrightarrow{G_0} & E \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ A & \xrightarrow{G_1} & B \end{array}$$

By general results of Lawvere A-theories, **GLTS**-algebras form a category **GLTS-Alg**. Arrows of **GLTS-Alg** are similar to those of $\mathbf{Cat}_o^{\rightarrow}$ strictly preserving the **GLTS**-structure. We write U for the forgetful functor from **GLTS-Alg** to $\mathbf{Cat}_o^{\rightarrow}$.

Similarly to Chapter 2, we consider a signature and an interpretation as an object and an arrow of $\mathbf{Cat}_o^{\rightarrow}$, respectively. Let **Data** be the category that has three objects **Pc**, **Test**, and **Com** and two arrows $\mathbf{pre}, \mathbf{post}\colon \mathbf{Com} \to \mathbf{Test}$. A signature $(P_{\mathbf{Pc}}, P_{\mathbf{Test}}, P_{\mathbf{Com}})$ gives rise to a functor into **Data** as follows. **Pred** is the discrete category given by the object set $P_{\mathbf{Pc}} + P_{\mathbf{Test}} + P_{\mathbf{Com}}$. The functor **type** from **Pred** to **Data** classifies symbols of $P_{\mathbf{Pc}} + P_{\mathbf{Test}} + P_{\mathbf{Com}}$.

We can now interpret labels by the same way as fibrational models of predicate logics.

Definition 3.3.2 (Interpretation of labels)**.** An *interpretation of labels* in a **GLTS**-algebra $p\colon E \to B$ is an arrow $G\colon \mathbf{type} \to Up$ in $\mathbf{Cat}_o^{\rightarrow}$.

With the signature $(P_{\mathbf{Pc}}, P_{\mathbf{Test}}, P_{\mathbf{Com}})$ of Section 3.1, a standard interpretation **intZ** of labels in the subset fibration is given as follows. To objects of **Data**, the corresponding sets are assigned by the functor $\mathbf{intZ}_1$. This functor specifies that there are four program counters and two integer variables.

$$\begin{array}{lcl} \mathbf{intZ}_1 & : & \mathbf{Data} \to \mathbf{Set} \\ \mathbf{intZ}_1(\mathbf{Pc}) & = & \{1,2,3,4\} \\ \mathbf{intZ}_1(\mathbf{Test}) & = & \mathbf{Z} \times \mathbf{Z} \\ \mathbf{intZ}_1(\mathbf{Com}) & = & \mathbf{Z} \times \mathbf{Z} \times \mathbf{Z} \times \mathbf{Z} \end{array}$$

$$\begin{array}{lcl} \mathbf{intZ}_1(\mathbf{pre})(x,y,x',y') & = & (x,y) \\ \mathbf{intZ}_1(\mathbf{post})(x,y,x',y') & = & (x',y') \end{array}$$

By the definition of $\mathbf{Cat}_o^{\rightarrow}$, an arrow from **type** to **base** needs one more functor $\mathbf{intZ}_0 \colon \mathbf{Pred} \to \mathbf{Sub}(\mathbf{Set})$ that satisfies the equation $\mathbf{base} \circ \mathbf{intZ}_0 = \mathbf{intZ}_1 \circ \mathbf{type}$. The equation means that, for $x \in \mathbf{Pred}$, $\mathbf{intZ}_0(x)$ is a subset of $\mathbf{intZ}_1(\mathbf{type}(x))$.

$$\begin{array}{lcl} \mathbf{intZ}_0(\mathbf{is1}) & = & (\{1\},\{1,2,3,4\}) \\ \mathbf{intZ}_0(\mathbf{is2}) & = & (\{2\},\{1,2,3,4\}) \\ \mathbf{intZ}_0(\mathbf{is3}) & = & (\{3\},\{1,2,3,4\}) \\ \mathbf{intZ}_0(\mathbf{is4}) & = & (\{4\},\{1,2,3,4\}) \\ \mathbf{intZ}_0(\mathbf{x >= 0}) & = & (\{(x,y) \mid 0 \le x\}, \mathbf{Z} \times \mathbf{Z}) \\ \mathbf{intZ}_0(\mathbf{x < 0}) & = & (\{(x,y) \mid x < 0\}, \mathbf{Z} \times \mathbf{Z}) \\ \mathbf{intZ}_0(\mathbf{y >= 0}) & = & (\{(x,y) \mid 0 \le y\}, \mathbf{Z} \times \mathbf{Z}) \\ \mathbf{intZ}_0(\mathbf{x = x + y}) & = & (\{(x,y,x',y') \mid x' = x + y, y' = y\}, \mathbf{Z} \times \mathbf{Z} \times \mathbf{Z} \times \mathbf{Z}) \\ \mathbf{intZ}_0(\mathbf{skip}) & = & (\{(x,y,x',y') \mid x' = x,\ y' = y\}, \mathbf{Z} \times \mathbf{Z} \times \mathbf{Z} \times \mathbf{Z}) \end{array}$$

Section 4.6 shows that the forgetful functor U from **GLTS-Alg** to $\mathbf{Cat}_o^{\rightarrow}$ has a left adjoint, hence, one can construct a free **GLTS**-algebra from an arbitrary functor. From $\mathbf{type} \colon \mathbf{Pred} \to \mathbf{Data}$, one can construct the free **GLTS**-algebra on **type** that includes all GLTS-formulas as objects of a fibre. That is why an interpretation of labels extends to an interpretation of all GLTS-formulas.

Definition 3.3.3 (Interpretation of GLTS-formula)**.** Let $G \colon \mathbf{type} \to Uq$ be an interpretation of labels into **GLTS**-algebra $q \colon D \to A$. To an arbitrary GLTS-formula R, we define an object $\overline{G}_0(R)$ of the fibre $D_{G_1(\mathbf{Pc}) \times G_1(\mathbf{Pc}) \times G_1(\mathbf{Com})}$ as follows.

- $\overline{G}_0(\bot)$ is the fibred initial object.
- $\overline{G}_0(\top)$ is the fibred terminal object.
- $\overline{G}_0(R_1 \vee R_2)$ is the fibred coproduct of $\overline{G}_0(R_1)$ and $\overline{G}_0(R_2)$.
- $\overline{G}_0(R_1 \wedge R_2)$ is the fibred product of $\overline{G}_0(R_1)$ and $\overline{G}_0(R_2)$.
- $\overline{G}_0(\mathbf{From}(p))$ is $u^*(G_0(p))$ where u is the projection from $(G_0(\mathbf{Pc}) \times G_0(\mathbf{Pc})) \times G_0(\mathbf{Com})$ to the left $G_0(\mathbf{Pc})$.
- $\overline{G}_0(\mathbf{To}(p))$ is $u^*(G_0(p))$ where u is the projection from $(G_1(\mathbf{Pc}) \times G_1(\mathbf{Pc})) \times G_1(\mathbf{Com})$ to the right $G_1(\mathbf{Pc})$.
- $\overline{G}_0(\mathbf{If}(t))$ is $u^*(G_1\mathbf{pre})^*(G_0(t))$ where u is the projection from $(G_1(\mathbf{Pc}) \times G_1(\mathbf{Pc})) \times G_1(\mathbf{Com})$ to $G_1(\mathbf{Com})$.
- $\overline{G}_0(\mathbf{Do}(c))$ is $u^*(G_0(c))$ where u is the projection from $(G_1(\mathbf{Pc}) \times G_1(\mathbf{Pc})) \times G_1(\mathbf{Com})$ to $G_1(\mathbf{Com})$.

For GLTS-formula **Loopxy** of Section 3.1 and interpretation **intZ** of labels, $\overline{\mathbf{intZ}}_0(\mathbf{Loopxy})$ is the following subset of $\{1,2,3,4\} \times \{1,2,3,4\} \times \mathbf{Z} \times \mathbf{Z} \times \mathbf{Z} \times \mathbf{Z}$.

$$\begin{aligned}
&\{(1,2,a,b,a,b) \mid 0 \le a\} \cup \\
&\{(1,4,a,b,a,b) \mid a < 0\} \cup \\
&\{(2,3,a,b,a',b) \mid a + b = a'\} \cup \\
&\{(3,2,a,b,a,b) \mid 0 \le a\} \cup \\
&\{(3,4,a,b,a,b) \mid a < 0\}
\end{aligned}$$

This is the standard interpretation intended for the program in Section 3.1.

3.4 Abstraction between Interpretations of Labels

This section gives the notion of abstraction for interpretations of labels. It is based on the similar 2-categorical framework to abstraction of $R\mu$, that is, the enrichment of the set $\mathbf{Cat}_o^{\rightarrow}(\mathbf{type}, Up)$ to a category.

$\mathbf{Cat}^{\rightarrow}$ is the 2-category given by the following 2-cells with the same objects and arrows as $\mathbf{Cat}_o^{\rightarrow}$. For arrows $G, H \colon q \to p$, a 2-cell $\alpha \colon G \Rightarrow H$ consists of a natural transformation $\alpha_0\colon G_0 \Rightarrow H_0$ and a natural transformation $\alpha_1\colon G_1 \Rightarrow H_1$ satisfying $p\alpha_0 = \alpha_1 q$.

$$\begin{array}{ccc} & G_0 & \\ D & \xrightarrow[\;H_0\;]{\Downarrow\,\alpha_0} & E \\ \downarrow q & & \downarrow p \\ & G_1 & \\ A & \xrightarrow[\;H_1\;]{\Downarrow\,\alpha_1} & B \end{array}$$

Definition 3.4.1 (abstraction)**.** For **GLTS**-algebra q and interpretations of labels $G, H\colon \mathbf{type} \to Uq$, a 2-cell from G to H in $\mathbf{Cat}^{\to}$ is called an *abstraction*.

So, for **GLTS**-algebra $q\colon D \to A$ and interpretations of labels $G, H\colon \mathbf{type} \to Uq$, an abstraction $\gamma\colon G \Rightarrow H$ consists of

- $\gamma_1(\mathbf{Pc})\colon G_1(\mathbf{Pc}) \to H_1(\mathbf{Pc})$ in A;
- $\gamma_1(\mathbf{Test})\colon G_1(\mathbf{Test}) \to H_1(\mathbf{Test})$ in A;
- $\gamma_1(\mathbf{Com})\colon G_1(\mathbf{Com}) \to H_1(\mathbf{Com})$ in A satisfying $H_1\mathbf{pre} \circ \gamma_1(\mathbf{Com}) = \gamma_1(\mathbf{Test}) \circ G_1\mathbf{pre}$ and $H_1\mathbf{post} \circ \gamma_1(\mathbf{Com}) = \gamma_1(\mathbf{Test}) \circ G_1\mathbf{post}$;
- $\gamma_0(p)\colon G_0(p) \to H_0(p)$ in D satisfying $q\gamma_0(p) = \gamma_1(\mathbf{Pc})$ for $p \in P_{\mathbf{Pc}}$;
- $\gamma_0(t)\colon G_0(t) \to H_0(t)$ in D satisfying $q\gamma_0(t) = \gamma_1(\mathbf{Test})$ for $t \in P_{\mathbf{Test}}$; and
- $\gamma_0(c)\colon G_0(c) \to H_0(c)$ in D satisfying $q\gamma_0(c) = \gamma_1(\mathbf{Com})$ for $c \in P_{\mathbf{Com}}$.

We do not have to give γ_0 and H_0 directly. By the next theorem, an abstraction γ and an abstract interpretation H of labels are constructed from a concrete interpretation G of labels and γ_1. We prove this theorem for a general q instead of **type**.

Theorem 3.4.2. Let $p\colon E \to B$ be a cocomplete fibration. Let $q\colon D \to A$ be a functor. Let G be an arrow from q to Up in $\mathbf{Cat}^{\to}$. Then, a functor $K\colon A \to B$ and a natural transformation $\beta\colon G_1 \to K$ extend to an arrow $H\colon q \to Up$ and a 2-cell $\gamma\colon G \to H$ satisfying $H_1 = K$ and $\gamma_1 = \beta$.

Proof. (sketch) For object d in D, put $H_0 d = \Sigma_{\beta(qd)} G_0 d$. For arrow $f\colon d \to d'$ in D, $H_0 f\colon H_0 d \to H_0 d'$ is determined by the 2-categorical structure of cocomplete fibration (see Section 4.6.4) for the following arrows.

$$\begin{array}{ccccc} G_1 qd & \xrightarrow{\beta(qd)} & Kqd & & G_0 d \\ {\scriptstyle pG_0 f = G_1 qf}\downarrow & & \downarrow{\scriptstyle Kqf} & & \downarrow{\scriptstyle G_0 f} \\ G_1 qd' & \xrightarrow{\beta(qd')} & Kqd' & & G_0 d' \end{array}$$

For object d in D, $\gamma_0(d)\colon G_0 d \to H_0 d$ is determined as follows.

$$G_0 d \xrightarrow{\eta_{\beta(qd)}(G_0 d)} (\beta(qd))^* \Sigma_{\beta(qd)} G_0 d = (\beta(qd))^* H_0 d \xrightarrow{\overline{\beta(qd)}(H_0 d)} H_0 d$$

□

We will show an example of this construction. Put $p = \mathbf{base}\colon \mathbf{Sub}(\mathbf{Set}) \to \mathbf{Set}$, $q = \mathbf{type}\colon \mathbf{Pred} \to \mathbf{Data}$, and $G = \mathbf{intZ}\colon \mathbf{type} \to U\mathbf{base}$. Let K be the following functor.

$$\begin{array}{rcl} K & : & \mathbf{Data} \to \mathbf{Set} \\ K(\mathbf{Pc}) & = & \{1, 2, 3, 4\} \\ K(\mathbf{Test}) & = & \mathbf{Bool} \times \mathbf{Bool} \\ K(\mathbf{Com}) & = & \mathbf{Bool} \times \mathbf{Bool} \times \mathbf{Bool} \times \mathbf{Bool} \\ K(\mathbf{pre})(x, y, x', y') & = & (x, y) \\ K(\mathbf{post})(x, y, x', y') & = & (x', y') \end{array}$$

The sets of program counters assigned by $\mathbf{intZ}$ and K are equal. Although $\mathbf{intZ}_1$ considers two variables x and y as integer variables, K considers them as boolean variables. The two different value domains are connected by the natural transformation $\beta\colon \mathbf{intZ}_1 \to K$ induced by the function $\mathbf{pos}$.

$$\begin{array}{rcl} \mathbf{pos} & : & \mathbf{Z} \to \mathbf{Bool} = \{\mathbf{t}, \mathbf{f}\} \\ \mathbf{pos}(x) & = & \begin{cases} \mathbf{t} & (0 \le x) \\ \mathbf{f} & (x < 0) \end{cases} \end{array}$$

$$
\begin{aligned}
\beta(\mathbf{Pc}) &: \mathbf{intZ}_1(\mathbf{Pc}) \to K(\mathbf{Pc}) \\
\beta(\mathbf{Pc})(c) &= c \\
\beta(\mathbf{Test}) &: \mathbf{intZ}_1(\mathbf{Test}) \to K(\mathbf{Test}) \\
\beta(\mathbf{Test})(x, y) &= (\mathbf{pos}(x), \mathbf{pos}(y)) \\
\beta(\mathbf{Com}) &: \mathbf{intZ}_1(\mathbf{Com}) \to K(\mathbf{Com}) \\
\beta(\mathbf{Com})(x, y, x', y') &= (\mathbf{pos}(x), \mathbf{pos}(y), \mathbf{pos}(x'), \mathbf{pos}(y'))
\end{aligned}
$$

By Theorem 3.4.2, K and β automatically extend to an arrow $\mathbf{intB}\colon \mathbf{type} \to U\mathbf{base}$ and an abstraction $\mathbf{absZB}\colon \mathbf{intZ} \to \mathbf{intB}$ such that $\mathbf{intB}_1 = K$ and $\mathbf{absZB}_1 = \beta$. The interpretation $\mathbf{intB}$ assigns the following subset of the set $K(\mathbf{Com})$ to the symbol $\mathbf{x} = \mathbf{x} + \mathbf{y}$.

$$\{(\mathbf{t}, \mathbf{t}, \mathbf{t}, \mathbf{t}), (\mathbf{t}, \mathbf{f}, \mathbf{t}, \mathbf{f}), (\mathbf{t}, \mathbf{f}, \mathbf{f}, \mathbf{f}), (\mathbf{f}, \mathbf{t}, \mathbf{t}, \mathbf{t}), (\mathbf{f}, \mathbf{t}, \mathbf{f}, \mathbf{t}), (\mathbf{f}, \mathbf{f}, \mathbf{f}, \mathbf{f})\}$$

This subset reflects that the sum of positive integers is positive, and so on. After all, the abstract interpretation of the command $\mathbf{x} = \mathbf{x} + \mathbf{y}$ is automatically constructed from the function **pos**.

By Definition 3.3.3, an interpretation of labels extends to an interpretation of GLTS-formulas. Similarly, an abstraction also extends to one between interpretations of GLTS-formulas.

Theorem 3.4.3 (Abstract interpretation of GLTS-formula)**.** For interpretations $G, H\colon \mathbf{type} \to Uq$ into **GLTS**-algebra $q\colon D \to A$, an abstraction $\gamma\colon G \Rightarrow H$ extends to arrows $\overline{\gamma}_0(R)\colon \overline{G}_0(R) \to \overline{H}_0(R)$ in D satisfying $q\overline{\gamma}_0(R) = (\gamma_1(\mathbf{Pc}) \times \gamma_1(\mathbf{Pc})) \times \gamma_1(\mathbf{Com})$ for each GLTS-formula R.

This theorem is a corollary of the theorem that the forgetful functor U from **GLTS-Alg** to $\mathbf{Cat}_o^{\to}$ extends to a right 2-adjoint (See Section 4.6.4).

As an example, we will show **absZB** and **Loopxy** again. By the definition of arrows of $\mathbf{Sub}(\mathbf{Set})$, for $i, j \in \{1, 2, 3, 4\}$ and $x, y, x', y' \in \mathbf{Z}$, if $(i, j, x, y, x', y') \in \overline{\mathbf{intZ}}_0(\mathbf{Loopxy})$, then $(i, j, \mathbf{pos}(x), \mathbf{pos}(y), \mathbf{pos}(x'), \mathbf{pos}(y')) \in \overline{\mathbf{intB}}_0(\mathbf{Loopxy})$.

3.5 From GLTS to $R\mu$

In this section, an interpretation of a GLTS-formula is translated into an interpretation of $R\mu$.

For example, Section 3.3 gives the standard interpretation of this program as an interpretation of GLTS-formula **Loopxy**.

```
/* 1 */
while(0 =< x){
    /* 2 */
    x = x+y;
    /* 3 */
}
/* 4 */
```

To formulate this program as an interpretation of $R\mu$, we take the following signature **Atm** and the theory Δ.

$$\begin{aligned} \mathbf{Atm} &= \{\mathbf{isn't1},\ \mathbf{isn't4},\ (\mathbf{x} < \mathbf{0}),\ (\mathbf{y} < \mathbf{0}), \square\} \\ \Delta &= \emptyset \end{aligned}$$

Let $p\colon E \to B$ be a **GLTS**-algebra, a complete fibration, and a fibred CCC with fibred initial object $\bot$. For the signature $(P_{\mathbf{Pc}}, P_{\mathbf{Test}}, P_{\mathbf{Com}})$ in Section 3.1, an interpretation $G\colon \mathbf{type} \to Up$ of labels extends to the following interpretation $\overline{G}'\colon \Sigma_{\mathbf{Atm}} \to \mathbf{Pos_{CL}}$ of $R\mu$. Here, $\lambda_{x,y}$ and $\rho_{x,y}$ are the left projection and the right projection from $x \times y$, respectively.

$$\begin{aligned}
\overline{G}'\, * &= E_{G_1(\mathbf{Pc}) \times G_1(\mathbf{Test})} \\
\overline{G}'\, \square &= \Pi_{\lambda_{G_1(\mathbf{Pc}),G_1(\mathbf{Pc})} \times G_1(\mathbf{pre})} \circ [\overline{G}_0 \mathbf{Loopxy}, -] \circ (\rho_{G_1(\mathbf{Pc}),G_1(\mathbf{Pc})} \times G_1(\mathbf{post}))^* \\
\overline{G}'\, \mathbf{isn't1} &= X \mapsto [\lambda^*_{G_1(\mathbf{Pc}),G_1(\mathbf{Test})}(G_0(\mathbf{is1})), \bot] \\
\overline{G}'\, \mathbf{isn't4} &= X \mapsto [\lambda^*_{G_1(\mathbf{Pc}),G_1(\mathbf{Test})}(G_0(\mathbf{is4})), \bot] \\
\overline{G}'\, (\mathbf{x} < \mathbf{0}) &= X \mapsto [\rho^*_{G_1(\mathbf{Pc}),G_1(\mathbf{Test})}(G_0(\mathbf{x} >= \mathbf{0})), \bot] \\
\overline{G}'\, (\mathbf{y} < \mathbf{0}) &= X \mapsto [\rho^*_{G_1(\mathbf{Pc}),G_1(\mathbf{Test})}(G_0(\mathbf{y} >= \mathbf{0})), \bot]
\end{aligned}$$

When we put $p = \mathbf{Sub}(\mathbf{Set})$, the interpretation $\overline{\mathbf{intZ}}'$ for $\mathbf{intZ}$ in Section 3.1 of-course match the intended semantics of the program we want to verify.

We also show the translation from an abstraction for GLTS-formulas into an abstraction for $R\mu$. Let $p\colon E \to B$ be a fibred CCC. For $f\colon \psi \to \varphi$ satisfying $pf = u\colon \Delta \to \Gamma$, there exists a natural transformation from $u^* \circ [\varphi, -]\colon E_\Gamma \to E_\Delta$

to $[\psi, -] \circ u^* \colon E_\Gamma \to E_\Delta$. By using the fact as a lemma, we will prove the next theorem.

Theorem 3.5.1. Let $p \colon E \to B$ be a **GLTS**-algebra that is a complete fibred CCC. For $\gamma \colon G \Rightarrow H \colon \mathbf{type} \to Up$, there exists the following abstraction $\gamma' \colon \overline{G}' \Rightarrow \overline{H}' \colon \Sigma_{\mathbf{Atm}} \to \mathbf{Pos_{CL}}$ of $R\mu$.

$$\gamma'_* = \Pi_{\gamma_1(\mathbf{Pc}) \times \gamma_1(\mathbf{Test})}$$

Proof. The left adjoint of γ'_* is $(\gamma_1(\mathbf{Pc}) \times \gamma_1(\mathbf{Test}))^*$. The lax naturality for $\square$ holds by the following diagram. Here, $g = \rho_{G_1(\mathbf{Pc}),G_1(\mathbf{Pc})} \times G_1(\mathbf{post})$ and $h = \rho_{H_1(\mathbf{Pc}),H_1(\mathbf{Pc})} \times H_1(\mathbf{post})$.

$$\begin{array}{ccc}
\overline{G}' * & \xrightarrow{\Pi_{\gamma_1(\mathbf{Pc}) \times \gamma_1(\mathbf{Test})}} & \overline{H}' * \\
\Big\uparrow \Pi_{\lambda_{G_1(\mathbf{Pc}),G_1(\mathbf{Pc})} \times G_1(\mathbf{pre})} & \geq & \Pi_{\lambda_{H_1(\mathbf{Pc}),H_1(\mathbf{Pc})} \times H_1(\mathbf{pre})} \Big\uparrow \\
E_{(G_1(\mathbf{Pc}) \times G_1(\mathbf{Pc})) \times G_1(\mathbf{Com})} & \xrightarrow{\Pi_{(\gamma_1(\mathbf{Pc}) \times \gamma_1(\mathbf{Pc})) \times \gamma_1(\mathbf{Com})}} & E_{(H_1(\mathbf{Pc}) \times H_1(\mathbf{Pc})) \times H_1(\mathbf{Com})} \\
\Big\uparrow [\overline{G}_0\mathbf{Loopxy}, -] & \geq & [\overline{H}_0\mathbf{Loopxy}, -] \Big\uparrow \\
E_{(G_1(\mathbf{Pc}) \times G_1(\mathbf{Pc})) \times G_1(\mathbf{Com})} & \xrightarrow{\Pi_{(\gamma_1(\mathbf{Pc}) \times \gamma_1(\mathbf{Pc})) \times \gamma_1(\mathbf{Com})}} & E_{(H_1(\mathbf{Pc}) \times H_1(\mathbf{Pc})) \times H_1(\mathbf{Com})} \\
\Big\uparrow g^* & \geq & h^* \Big\uparrow \\
\overline{G}' * & \xrightarrow[\Pi_{\gamma_1(\mathbf{Pc}) \times \gamma_1(\mathbf{Test})}]{} & \overline{H}' *
\end{array}$$

The lax naturality for $\mathbf{isn't1}$ holds by the following diagram.

$$\begin{array}{ccccc}
\overline{G}' * & \xrightarrow{\bot} & \overline{G}' * & \xrightarrow{[\lambda^*_{G_1(\mathbf{Pc}),G_1(\mathbf{Test})}(G_0(\mathbf{is1})), -]} & \overline{G}' * \\
\Big\downarrow & \leq & \Big\downarrow \Pi_{\gamma_1(\mathbf{Pc}) \times \gamma_1(\mathbf{Test})} & \leq & \Pi_{\gamma_1(\mathbf{Pc}) \times \gamma_1(\mathbf{Test})} \Big\downarrow \\
\overline{H}' * & \xrightarrow[\bot]{} & \overline{H}' * & \xrightarrow[{[\lambda^*_{H_1(\mathbf{Pc}),H_1(\mathbf{Test})}(H_0(\mathbf{is1})), -]}]{} & \overline{H}' *
\end{array}$$

Similarly, it is lax natural for $\mathbf{isn't4}$, $(\mathbf{x} < \mathbf{0})$, and $(\mathbf{y} < \mathbf{0})$. $\square$

We aim to show that the line 4 in the program is not reachable if x and y are positive in the initial line 1.

Put $p = \mathbf{Sub}(\mathbf{Set})$ and take $\mathbf{intZ}$ in Section 3.1. It is the concrete interpretation of labels. By Section 3.4, we get the abstraction $\mathbf{absZB} \colon \mathbf{intZ} \Rightarrow \mathbf{intB}$. By

Theorem 3.5.1, they extend to the abstraction $\overline{\mathbf{absZB}}'$ between interpretations $\overline{\mathbf{intZ}}'$ and $\overline{\mathbf{intB}}'$ of $R\mu$.

$$\begin{array}{lcl}
\overline{\mathbf{intZ}}'* & = & \wp(\{1,2,3,4\} \times \mathbf{Z} \times \mathbf{Z}) \\
\overline{\mathbf{intB}}'* & = & \wp(\{1,2,3,4\} \times \mathbf{Bool} \times \mathbf{Bool}) \\
\overline{\mathbf{absZB}}'_* & : & \wp(\{1,2,3,4\} \times \mathbf{Z} \times \mathbf{Z}) \to \wp(\{1,2,3,4\} \times \mathbf{Bool} \times \mathbf{Bool}) \\
\overline{\mathbf{absZB}}'_*(X) & = & \{(c,a,b) \mid \forall x \in \mathbf{Z}. \forall y \in \mathbf{Z}. (a = \mathbf{pos}(x) \wedge b = \mathbf{pos}(y)) \Rightarrow (c,x,y) \in X\} \\
\mathbf{left}_* & : & \wp(\{1,2,3,4\} \times \mathbf{Bool} \times \mathbf{Bool}) \to \wp(\{1,2,3,4\} \times \mathbf{Z} \times \mathbf{Z}) \\
\mathbf{left}_*(X) & = & \{(c,x,y) \mid (c, \mathbf{pos}(x), \mathbf{pos}(y)) \in X\}
\end{array}$$

The arrow $\mathbf{left}_*$ is a left adjoint of $\overline{\mathbf{absZB}}'_*$.

The safety property we want to show can be formally stated as the $R\mu$-formula σ:

$$\mathbf{isn't1} \vee (\mathbf{x} < \mathbf{0}) \vee (\mathbf{y} < \mathbf{0}) \vee \nu(\mathbf{isn't4}, \Box)$$

To show that the property holds is to show that $[\![\sigma]\!]_{\overline{\mathbf{intZ}}'}$ is the greatest element of $\mathbf{Pos_{CL}}(\overline{\mathbf{intZ}}'*, \overline{\mathbf{intZ}}'*)$. However, we can not directly check if $s \in [\![\sigma]\!]_{\overline{\mathbf{intZ}}'}(\cdot)$ for each $s \in \{1,2,3,4\} \times \mathbf{Z} \times \mathbf{Z}$ as $\{1,2,3,4\} \times \mathbf{Z} \times \mathbf{Z}$ is infinite.

Now it is directly checkable whether $[\![\sigma]\!]_{\overline{\mathbf{intB}}'}$ is the greatest in $\mathbf{Pos_{CL}}(\overline{\mathbf{intB}}'*, \overline{\mathbf{intB}}'*)$ by checking every element of finite $\{1,2,3,4\} \times \mathbf{Bool} \times \mathbf{Bool}$. By the formula-preservation theorem (Corollary 2.5.7), the formula σ satisfies $[\![\sigma]\!]_{\overline{\mathbf{intB}}'} \circ \overline{\mathbf{absZB}}'_* \leq \overline{\mathbf{absZB}}'_* \circ [\![\sigma]\!]_{\overline{\mathbf{intZ}}'}$, which is equivalent to $\mathbf{left}_* \circ [\![\sigma]\!]_{\overline{\mathbf{intB}}'} \circ \overline{\mathbf{absZB}}'_* \leq [\![\sigma]\!]_{\overline{\mathbf{intZ}}'}$ in our case. By the definition of $\mathbf{left}_*$, if the $[\![\sigma]\!]_{\overline{\mathbf{intB}}'}$ is the greatest, then so is $[\![\sigma]\!]_{\overline{\mathbf{intZ}}'}$ as desired.

Now, we compute $[\![\nu(\mathbf{isn't4}, \Box)]\!]_{\overline{\mathbf{intB}}'}$. By the structure of $\mathbf{Pos_{CL}}$, $[\![\nu(\mathbf{isn't4}, \Box)]\!]_{\overline{\mathbf{intB}}'}(\cdot)$ is the greatest fixed point of the following function $F\colon \wp(\{1,2,3,4\} \times \mathbf{Bool} \times \mathbf{Bool}) \to \wp(\{1,2,3,4\} \times \mathbf{Bool} \times \mathbf{Bool})$.

$$F(X) = [\![\mathbf{isn't4}]\!]_{\overline{\mathbf{intB}}'}(X) \cap [\![\Box]\!]_{\overline{\mathbf{intB}}'}(X)$$

Since $\{1,2,3,4\} \times \mathbf{Bool} \times \mathbf{Bool}$ is a finite set, we can compute the value as follows. Since $F^4(\{1,2,3,4\} \times \mathbf{Bool} \times \mathbf{Bool}) = F^5(\{1,2,3,4\} \times \mathbf{Bool} \times \mathbf{Bool})$, the greatest fixed point $[\![\nu(\mathbf{isn't4}, \Box)]\!]_{\overline{\mathbf{intB}}'}(\cdot)$ is $F^5(\{1,2,3,4\} \times \mathbf{Bool} \times \mathbf{Bool}) =$

$\{(1,\mathbf{t},\mathbf{t}),(2,\mathbf{t},\mathbf{t}),(3,\mathbf{t},\mathbf{t})\}$.

$$
\begin{aligned}
[\![\Box]\!]_{\overline{\mathbf{intB}}'}(X) \;=\; & (\{4\}\times\{\mathbf{t},\mathbf{f}\}\times\{\mathbf{t},\mathbf{f}\})\cup\\
& \{(1,\mathbf{f},b)\mid b\in\{\mathbf{t},\mathbf{f}\},(4,\mathbf{f},b)\in X\}\cup\\
& \{(1,\mathbf{t},b)\mid b\in\{\mathbf{t},\mathbf{f}\},(2,\mathbf{t},b)\in X\}\cup\\
& \{(2,a,b)\mid a,b\in\{\mathbf{t},\mathbf{f}\},(3,a,b),(3,b,b)\in X\}\cup\\
& \{(3,\mathbf{f},b)\mid b\in\{\mathbf{t},\mathbf{f}\},(4,\mathbf{f},b)\in X\}\cup\\
& \{(3,\mathbf{t},b)\mid b\in\{\mathbf{t},\mathbf{f}\},(2,\mathbf{t},b)\in X\}
\end{aligned}
$$

$$
\begin{aligned}
F^0(\{1,2,3,4\}\times\mathbf{Bool}\times\mathbf{Bool}) &= \{1,2,3,4\}\times\mathbf{Bool}\times\mathbf{Bool}\\
F^1(\{1,2,3,4\}\times\mathbf{Bool}\times\mathbf{Bool}) &= \{1,2,3\}\times\{\mathbf{t},\mathbf{f}\}\times\{\mathbf{t},\mathbf{f}\}\\
F^2(\{1,2,3,4\}\times\mathbf{Bool}\times\mathbf{Bool}) &= \{(1,\mathbf{t},\mathbf{t}),(1,\mathbf{t},\mathbf{f}),(2,\mathbf{t},\mathbf{t}),(2,\mathbf{t},\mathbf{f}),(2,\mathbf{f},\mathbf{t}),\\
&\quad (2,\mathbf{f},\mathbf{f}),(3,\mathbf{t},\mathbf{t}),(3,\mathbf{t},\mathbf{f})\}\\
F^3(\{1,2,3,4\}\times\mathbf{Bool}\times\mathbf{Bool}) &= \{(1,\mathbf{t},\mathbf{t}),(1,\mathbf{t},\mathbf{f}),(2,\mathbf{t},\mathbf{t}),(3,\mathbf{t},\mathbf{t}),(3,\mathbf{t},\mathbf{f})\}\\
F^4(\{1,2,3,4\}\times\mathbf{Bool}\times\mathbf{Bool}) &= \{(1,\mathbf{t},\mathbf{t}),(2,\mathbf{t},\mathbf{t}),(3,\mathbf{t},\mathbf{t})\}\\
F^5(\{1,2,3,4\}\times\mathbf{Bool}\times\mathbf{Bool}) &= \{(1,\mathbf{t},\mathbf{t}),(2,\mathbf{t},\mathbf{t}),(3,\mathbf{t},\mathbf{t})\}
\end{aligned}
$$

Therefore, we can easily prove that $[\![\sigma]\!]_{\overline{\mathbf{intB}}'}(\cdot)=\{1,2,3,4\}\times\mathbf{Bool}\times\mathbf{Bool}$ for the $R\mu$-formula σ.

3.6 Functional Simulations

In this section, we compare various simulations.

By Section 2.5, simulations are examples of abstraction in $R\mu$. In general, a simulation is a binary relation between sets of states. However, there are simulations which are also functions, for example, in the framework of abstract-data mapping and predicate abstraction. They are examples of abstraction between interpretation of labels.

In *abstract-data mapping* [9, 8], a abstraction function assigns an abstract data domain to each variable in programs. For example, the following functions are abstraction functions.

- Elements of this abstract domain represent positive or not.

$$\begin{aligned}\mathbf{pos} &: \mathbf{Z} \to \mathbf{Bool} = \{\mathbf{t}, \mathbf{f}\}\\ \mathbf{pos}(x) &= \begin{cases}\mathbf{t} & (0 \le x)\\ \mathbf{f} & (x < 0)\end{cases}\end{aligned}$$

- Elements of this abstract domain represent the signs.

$$\begin{aligned}\mathbf{sign} &: \mathbf{Z} \to \{\mathbf{posi}, \mathbf{zero}, \mathbf{nega}\}\\ \mathbf{sign}(x) &= \begin{cases}\mathbf{posi} & (0 < x)\\ \mathbf{zero} & (x = 0)\\ \mathbf{nega} & (x < 0)\end{cases}\end{aligned}$$

- Elements of this abstract domain represent comparison of the integer with -1, 0, and 1.

$$\begin{aligned}\mathbf{range} &: \mathbf{Z} \to \{\le \mathbf{-2}, \mathbf{-1}, \mathbf{0}, \mathbf{1}, \mathbf{2} \le\}\\ \mathbf{range}(x) &= \begin{cases}\le \mathbf{-2} & (x \le -2)\\ \mathbf{-1} & (x = -1)\\ \mathbf{0} & (x = 0)\\ \mathbf{1} & (x = 1)\\ \mathbf{2} \le & (2 \le x)\end{cases}\end{aligned}$$

- This abstract domain represent that the variable is ignored. (e.g. in cone of influence reduction)

$$\begin{aligned}\mathbf{ignore} &: A \to \{*\}\\ \mathbf{ignore}(x) &= *\end{aligned}$$

Abstract-data mapping sometimes deals with a tuple of plural variables as one variable.

- This abstract domain is constructed by the equivalence relation $(x, y) = (y, x)$.

$$\begin{aligned}\mathbf{sym} &: S \times S \to \{\{x, y\} \mid x, y \in S\}\\ \mathbf{sym}(x, y) &= \{x, y\}\end{aligned}$$

In the framework of *predicate abstraction* [13], one takes predicates $\varphi_1, \ldots, \varphi_n$ on the state set S. They assign $\mathbf{Bool}^n$ to the abstract state set of programs. For example, the following predicates are used in predicate abstraction.

- This predicate represents whether the value of the unique integer variable is positive or not.

$$\begin{aligned} \mathbf{pos} &: \mathbf{Z} \to \mathbf{Bool} \\ \mathbf{pos}(x) &= \begin{cases} \mathbf{t} & (0 \leq x) \\ \mathbf{f} & (x < 0) \end{cases} \end{aligned}$$

- This predicate compares plural integer variables.

$$\begin{aligned} \mathbf{less} &: \mathbf{Z} \times \mathbf{Z} \to \mathbf{Bool} \\ \mathbf{less}(x, y) &= \begin{cases} \mathbf{t} & (x < y) \\ \mathbf{f} & (o.w.) \end{cases} \end{aligned}$$

Abstractions in $R\mu$ also include simulations which are not functions. We show two examples of them. Let interpretation m correspond to a Kripke model (S, R, Q) and m' correspond to a Kripke model (S', R', Q') by Example 2.3.5. By Example 2.5.2, a simulation $\rho \subseteq S \times S'$ gives rise to an abstraction γ from m to m' such that $\gamma_*(X) = \{s' \in S' \mid \forall s \in S.(s, s') \in \rho \Rightarrow s \in X\}$.

A partial function is one example. Let $\mathbf{List}(A)$ be the set of all finite lists of elements of A. Putting $S = \mathbf{List}(A)$ and $S' = A$, we define ρ such that $(l, a) \in \rho$ if and only if the head element of l is a. By the above construction, we can define γ_* from the relation ρ. Moreover, by Theorem 2.5.8, we can construct an abstract interpretation from a concrete interpretation. However, ρ is not a function. Therefore, ρ is not a direct example of abstraction between interpretation of labels in this chapter.

A pair of functions is the other example. Let S' be the direct sum $A + B = \{(0, a) \mid a \in A\} \cup \{(1, b) \mid b \in B\}$ of sets A and B. For functions $f\colon S \to A$ and $g\colon S \to B$, we can define ρ by $\{(s, (0, f(s))) \mid s \in S\} \cup \{(s, (1, g(s))) \mid s \in S\}$. However, ρ is not a function.

Chapter 4

Algebraic Structures on Enriched Categories

The correspondence between finitary monads on **Set** and finitary algebraic theories (i.e., collections of basic operations and equations such that the arity of each basic operation is finite) is one of the deepest relationships in category theory [44]. Given a finitary algebraic theory, a term model of the theory is a free algebra of the corresponding finitary monad. The correspondence is a framework to give semantics of various logics or programming languages.

The notion of finitary algebraic theory is formulated as Lawvere theory [3]. The Lawvere theory corresponding to a finitary monad is invariant, up to the simple and obvious notion of isomorphism between them. In the paper [37, 38], Nishizawa and Power generalise the correspondence between Lawvere theories and finitary monads on **Set**. Ordinary Lawvere theories, more generally the enriched Lawvere theories of [42], are special cases, with the respective definitions of model agreeing.

In this chapter, we introduce the notion. We first choose a category V in which to enrich, and then we choose a base V-category A. We then define a notion that we call a *Lawvere A-theory* and we extend the above correspondence to one between Lawvere A-theories and finitary V-enriched monads on the V-category A. For instance, taking V to be **Set** and A to be the **Set**-enriched category (i.e., the ordinary category) $\mathbf{Cat}_o$, we can consider structure on $\mathbf{Cat}_o$. This allows us to capture structures that we could not capture when A was identified with V as in the past. For instance, we can consider cartesian closed structure in this setting, which was impossible before because of the contravariance involved with closedness [5].

The techniques we develop here may also help with sophisticated computational effects such as probabilistic nondeterminism, in which one considers the category of dcpo's as an ω**Cpo**-enriched category [15], but that requires further investigation of size. Our definition of Lawvere A-theory implies a simplified formulation of V-enriched algebraic structure on A: we do not give a precise formulation here, but we illustrate by example that our definition allows less complicated formulation of the equations.

4.1 Lawvere A-theories: Formulation of Algebraic Structures

We assume that V is locally finitely presentable as a symmetric monoidal closed category and that A is a locally finitely presentable V-category: symmetry of V is not necessary here, but it is convenient for exposition, includes all our leading examples, and corresponds to most of the relevant literature. The precise definitions of these notions can be found in [20, 19], but in order to understand the point of this paper, one only needs to know examples that appear in the computer science literature [4, 23, 44]. **Set** and **Cat** are locally finitely presentable as symmetric monoidal closed categories. Locally finitely presentable **Set**-categories are exactly ordinary locally finitely presentable categories, such as **Set**, $\mathbf{Set}^k$, **Poset**, $\mathbf{Cat}_o$, **LocOrd**, and $\mathbf{Cat}_o^{\rightarrow}$. **Cat** is a locally finitely presentable **Cat**-category, and that statement extends to V given axiomatically as above.

We write A_f for a skeleton of the full sub-V-category of A given by the finitely presentable objects of A, and we let $\iota\colon A_f \to A$ denote the inclusion V-functor. Following the canonical reference for enriched categories [20], we denote the composite V-functor

$$A \xrightarrow{Y} [A^{\mathrm{op}}, V] \xrightarrow{[\iota^{\mathrm{op}}, V]} [A_f^{\mathrm{op}}, V]$$

by $\tilde{\iota}$, where Y is an enriched version of the Yoneda embedding. For example, up to coherent isomorphism, the category $\mathbf{Set}_f$ is the category **Nat**, whose objects are natural numbers and whose arrows are all functions between them. The functor $\tilde{\iota}$ sends a set X to the functor $\mathbf{Set}(\iota-, X)$. For a more complex example, $(\mathbf{Cat}_o)_f$ is the category of finitely presentable categories, i.e., those categories that are freely generated on a finite graph or are given by coequalising a pair of functors between such freely generated categories.

We next need the idea of a finite cotensor. This generalises the notion of a finite power. A V-category A has *finite cotensors* if for every finitely presentable X in V and every Z in A, there exists an object Z^X of A together with a natural isomorphism

$$[X, A(-, Z)] \cong A(-, Z^X)$$

For example, in the case $V = \mathbf{Set}$, a finite cotensor means that X is finite and Z^X is a product of X copies of Z. In the case $A = V$, the cotensor Z^X is given by the exponential $[X, Z]$. We write $\mathbf{FC}(A, V)$ for the full sub-V-category of $[A, V]$ determined by those V-functors that preserve finite cotensors.

Finally, we need the notion of a *finite enriched limit*. The formal definition is complicated, so we shall not give it directly but rather use a characterisation theorem that makes the notion much easier to grasp [20]: a V-category admits all finite V-limits if and only if it admits all finite conical limits and all finite cotensors Z^X. Here, the notion of conical limit is exactly as one would expect, bearing in mind that enrichment means one wants an isomorphism in V between the object of cones over a digram and the homobject of comparison maps, rather than a mere bijection of sets [20]. We write $\mathbf{FL}(A, V)$ for the full sub-V-category of $[A, V]$ determined by those V-functors that preserve finite V-limits. The V-functor ι preserves all finite V-colimits, and representable V-functors preserve V-limits, so $\tilde{\iota}$ factors through $\mathbf{FL}(A_f^{\mathrm{op}}, V)$. So we sometimes consider $\tilde{\iota}$ as a V-functor from A to $\mathbf{FL}(A_f^{\mathrm{op}}, V)$. The central result of Gabriel-Ulmer duality, generalised to enriched categories, asserts that $\tilde{\iota}$ induces an equivalence $A \simeq \mathbf{FL}(A_f^{\mathrm{op}}, V)$ of V-categories [19]. Since $\mathbf{FL}(A_f^{\mathrm{op}}, V)$ is a full sub-V-category of $\mathbf{FC}(A_f^{\mathrm{op}}, V)$, we also sometimes consider $\tilde{\iota}$ as a V-functor from A to $\mathbf{FC}(A_f^{\mathrm{op}}, V)$.

Finally, we can write the central definition of this chapter. We assume V and A satisfy the axiomatic structure described above, i.e., A is a locally finitely presentable V-category for appropriate V.

Definition 4.1.1. A *Lawvere A-theory* is a small V-category L together with an identity-on-objects strict finite V-limit preserving V-functor $J\colon A_f^{\mathrm{op}} \to L$.

So the objects of L are exactly the objects of A_f^{op}. One understands them in this setting to be generalised *arities*, and one understands the arrows of L to be operations. This should become clearer when we study examples. But to see the distinction between preservation of limits and preservation of cotensors in our definition, consider the example of $V = \mathbf{Set}$ and $A = \mathbf{Cat}_o$, and note that the triangle category is a pushout in $(\mathbf{Cat}_o)_f$ constructed from two copies of the arrow category together with the unit category 1.

A map of Lawvere A-theories from L to L' is an identity-on-objects V-functor from L to L' that commutes with the V-functors from A_f^{op}. Together with the usual composition of V-functors, Lawvere A-theories and their maps yield an ordinary category we denote by $\mathbf{Law}_A$.

Definition 4.1.2. Given a Lawvere A-theory L with $J\colon A_f^{\mathrm{op}} \to L$, define its V-category of models by the following pullback in the category V-$\mathbf{Cat}$ of locally small V-categories.

$$\begin{array}{ccc} \mathbf{Mod}(L) & \xrightarrow{P_L} & [L, V] \\ {\scriptstyle U_L}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle [J,V]} \\ A & \xrightarrow[\tilde{\iota}]{} & [A_f^{\mathrm{op}}, V] \end{array}$$

We call objects of $\mathbf{Mod}(L)$ *models* of L.

So a model consists of an object X of A together with a functor $M\colon L \to V$ whose behaviour when restricted to A_f^{op} is completely determined by A. Thus a model is determined by X together with data and axioms arising from those maps in L that are not already in A_f^{op}.

There is a subtle 2-categorical point here that is particularly convenient for us. The pullback defining $\mathbf{Mod}(L)$ is unusual in that it is also a *bipullback* [5], meaning that if one systematically replaces equality of diagrams in V-$\mathbf{Cat}$ by coherent isomorphism, this pullback still satisfies the systematically weakened version of the universal property. That can readily be checked directly, but axiomatically, it holds because the V-functor $[J, V]$ satisfies an isomorphism lifting property. We shall henceforth largely gloss over this point for the sake of exposition.

It will be easier to explain examples and to characterise the definition in special cases if we first give an alternative definition of the V-category of models as provided by the following proposition.

Proposition 4.1.3. For any Lawvere A-theory L with $J\colon A_f^{\mathrm{op}} \to L$, the following diagram forms a pullback in V-**Cat**.

$$\begin{array}{ccc} \mathbf{Mod}(L) & \longrightarrow & \mathbf{FC}(L,V) \\ {\scriptstyle U_L}\downarrow & & \downarrow{\scriptstyle \mathbf{FC}(J,V)} \\ A & \xrightarrow[\tilde{\iota}]{} & \mathbf{FC}(A_f^{\mathrm{op}},V) \end{array}$$

Proof. First observe that L has finite cotensors: J is the identity on objects, so every object of L lies uniquely in the image of J; moreover, J strictly preserves finite cotensors, hence the result. Now note that the square

$$\begin{array}{ccc} \mathbf{FC}(L,V) & \xrightarrow{\text{inclusion}} & [L,V] \\ {\scriptstyle \mathbf{FC}(J,V)}\downarrow & & \downarrow{\scriptstyle [J,V]} \\ \mathbf{FC}(A_f^{\mathrm{op}},V) & \xrightarrow[\text{inclusion}]{} & [A_f^{\mathrm{op}},V] \end{array}$$

is a pullback: if M is a V-functor from L to V such that $M \circ J$ preserves finite V-cotensors, it follows from the above construction of pullbacks in L that M preserves them. The lemma now follows from the definition of $\mathbf{Mod}(L)$ and generalities about pullbacks. □

We now compare our definitions with those already in the literature. An ordinary Lawvere theory [3] is usually defined to be a small category L with finite products together with an identity-on-objects strict finite product preserving functor from $\mathbf{Nat}^{\mathrm{op}}$ to L. A model in **Set** is defined to be a finite product preserving functor from L to **Set**. Note that one assumes that L has finite products and that the functor from $\mathbf{Nat}^{\mathrm{op}}$ to L strictly preserves finite products, whereas in our general definition, we asked for strict preservation of finite limits but made no further assumption of existence of any kind of limits in L.

Theorem 4.1.4. An ordinary Lawvere theory is a Lawvere **Set**-theory and conversely. Moreover, the two definitions of the category of models agree.

Proof. Let L be any ordinary Lawvere theory. It corresponds to a finitary monad T. Moreover, L is isomorphic to the restriction of $\mathbf{Kl}(T)^{\mathrm{op}}$ to the natural numbers, and the functor $J\colon \mathbf{Nat}^{\mathrm{op}} \to L$ is given by the restriction of the canonical functor from $\mathbf{Set}$ to $\mathbf{Kl}(T)$. So $J\colon \mathbf{Nat}^{\mathrm{op}} \to L$ strictly preserves all finite limits of $\mathbf{Nat}$, as the corresponding finite colimits are strictly preserved both by the inclusion into $\mathbf{Set}$ and by the canonical functor into $\mathbf{Kl}(T)$. So every ordinary Lawvere theory is a Lawvere $\mathbf{Set}$-theory in the above sense. The converse is trivially true. For the statement about models, first observe that $\mathbf{Set}_f^{\mathrm{op}}$ is the free $\mathbf{Set}$-category with finite cotensors, i.e., finite powers, on 1. So $\tilde{\iota}$ yields a canonical equivalence $\mathbf{Set} \simeq \mathbf{FC}(\mathbf{Set}_f^{\mathrm{op}}, \mathbf{Set})$. So Proposition 4.1.3 implies $\mathbf{Mod}(L) \simeq \mathbf{FC}(L, \mathbf{Set})$ But all finite products on $\mathbf{Set}^{\mathrm{op}}$, hence also on L, are given by finite powers of copies of 1, i.e., by finite cotensors, and so preservation of finite powers is equivalent, in this setting, to preservation of finite products, hence the result. $\square$

Enriching this result, in [42], given V satisfying the axioms we have here, a Lawvere V-theory was defined to be a small V-category L with finite V-cotensors together with an identity-on-objects strict finite V-cotensor preserving V-functor $J\colon V_f^{\mathrm{op}} \to L$. The V-category of models of such a Lawvere V-theory was defined to be $\mathbf{FC}(L, V)$.

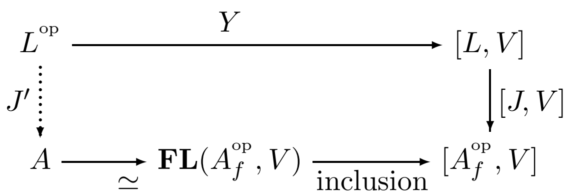
Theorem 4.1.5. If A is V, Lawvere A-theories are precisely Lawvere V-theories defined as above. Moreover, the two definitions of the V-category of models agree.

Proof. The proof of the correspondence is given by a simple enrichment of the proof of Theorem 4.1.4. Similarly for the statement about models. $\square$

4.2 Invariance of Lawvere A-Theories

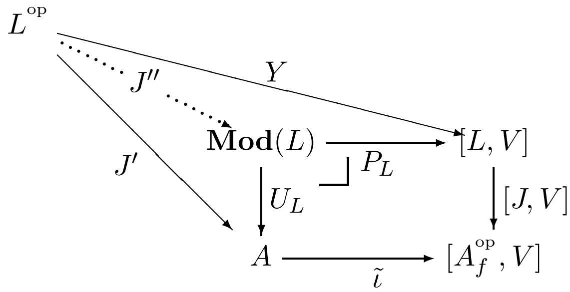
In this section, given any Lawvere A-theory, we prove that the forgetful V-functor $U_L\colon \mathbf{Mod}(L) \to A$ is finitarily V-monadic, yielding a finitary V-monad T_L on A. We further show how one can reconstruct L from T_L.

First observe that for any Lawvere A-theory L, since A is locally finitely presentable, so equivalent to $\mathbf{FL}(A_f^{\mathrm{op}}, V)$, and since representables preserve finite limits as does J, there is a canonical V-functor J' such that the following square commutes up to isomorphism:



One can make a slightly stronger statement: if one is willing to replace Y by a V-functor that is isomorphic to it, one can force the diagram actually to commute; although a minor point, that is convenient for us.

Applying the universal property determines a V-functor J'' as follows:



Since $\tilde{\iota}$ is fully faithful, so is P_L, and, since Y is also fully faithful, so is J''.

Proposition 4.2.1. For any Lawvere A-theory L and for any objects X of A_f and M of $\mathbf{Mod}(L)$, $\mathbf{Mod}(L)(J''J^{\text{op}}X, M)$ and $A(\iota X, U_L M)$ are V-naturally isomorphic in V.

Proof. By fully faithfulness of P_L, and by the enriched Yoneda lemma, with I the unit of V and since $L(JX, -) = P_L J'' J^{\text{op}} X$, and finally as $(P_L M)JX = ([J, V]P_L M)X = (\tilde{\iota} U_L M)X = A(\iota X, U_L M)$, we have the following string of V-natural correspondences:

$$\frac{\dfrac{\dfrac{J''J^{\text{op}}X \longrightarrow M}{P_L J'' J^{\text{op}} X \longrightarrow P_L M}}{I \longrightarrow (P_L M)JX}}{\iota X \longrightarrow U_L M}$$

□

Recall that U_L is defined as a pullback in V-**Cat**. So its defining diagram commutes exactly rather than just up to coherent isomorphism. That strictness is convenient, but we need care in order to maintain it. So, in the following, when we speak of a left Kan extension along a fully faithful inclusion V-functor, we shall assume that it is chosen to make the induced triangle commute exactly: a Kan extension along a fully faithful V-functor always makes the triangle commute up to coherent isomorphism [20], and when that V-functor is an inclusion, we can choose the Kan extension to make the triangle commute exactly.

Theorem 4.2.2. U_L has a left V-adjoint given by the left Kan extension of $J'' \circ J^{\mathrm{op}}$ along ι.

Proof. Let F_L be the left Kan extension of $J'' \circ J^{\mathrm{op}}$ along ι. It has a right adjoint H that sends a model M to $\mathbf{Mod}(L)(J''J^{\mathrm{op}}-, M)$. By Proposition 4.2.1, $HM \cong \mathbf{Mod}(L)(J''J^{\mathrm{op}}-, M) \cong A(\iota-, U_L M) \cong U_L M$ □

Theorem 4.2.3. U_L is finitary V-monadic.

Proof. By Theorem 4.2.2, U_L has a left V-adjoint. Let f, g be a U_L-split coequaliser pair in $\mathbf{Mod}(L)$. Since $[L, V]$ is cocomplete, $P_L f$ and $P_L g$ have a coequaliser, and the coequaliser can be chosen so that it is strictly preserved by $[J, V]$. Since a split coequaliser of $U_L f$ and $U_L g$ is also preserved by $\tilde{\iota}$, f and g have a coequaliser in $\mathbf{Mod}(L)$ and U_L strictly preserves it. So by Beck's monadicity theorem [3] and by remarks on enrichment of monadicity in [21], U_L is V-monadic. Finitariness of U_L follows from that of $[J, V]$ and $\tilde{\iota}$. □

We define T_L to be the finitary V-monad induced by a Lawvere A-theory L by Theorem 4.2.3. By the next corollary, we can reconstruct L from the monadic V-functor U_L.

Corollary 4.2.4. One rediscovers $(L^{\mathrm{op}}, J^{\mathrm{op}}, J'')$ as the (identity-on-objects,fully faithful) factorisation of $F_L \circ \iota$.

$$\begin{array}{ccc} L^{\mathrm{op}} & \xrightarrow{J''} & \mathbf{Mod}(L) \\ {\scriptstyle J^{\mathrm{op}}}\uparrow & & \uparrow{\scriptstyle F_L} \\ A_f & \xrightarrow[\iota]{} & A \end{array}$$

Proof. The diagram commutes by the construction of F_L in Theorem 4.2.2. Moreover, J^{op} is identity-on-objects and J'' is fully faithful. □

4.3 Lawvere A-Theories and Finitary V-Monads

In this section, we give an equivalence between the category of Lawvere A-theories and that of finitary V-monads on A. We first construct a Lawvere A-theory L_T from an arbitrary finitary V-monad T on A. We then show that the construction of Section 4.2 allows us to reconstruct T from L_T. Finally, we observe that the two constructions extend to an equivalence between the category of Lawvere A-theories and that of finitary V-monads on A.

For a finitary V-monad T on A, define (K_T, J_T, ι_T) by taking the (identity-on-objects,fully faithful) factorisation of $F^T \circ \iota$:

$$\begin{array}{ccc} K_T & \xrightarrow{\iota_T} & T\text{-}\mathbf{Alg} \\ {\scriptstyle J_T}\uparrow & & \uparrow{\scriptstyle F^T} \\ A_f & \xrightarrow[\iota]{} & A \end{array}$$

Since ι and F^T preserve finite V-colimits and ι_T reflects finite V-colimits, J_T is an identity-on-objects strict finite V-colimit preserving V-functor. So we define L_T to be K_T^{op}.

Note the similarity between this definition and Corollary 4.2.4. Also observe that, letting F_T be the canonical left V-adjoint from A to the Kleisli V-category $\mathbf{Kl}(T)$, we could equally have defined (K_T, J_T, ι_T) by taking the (identity-on-objects,fully faithful) factorisation of $F_T \circ \iota$:

$$\begin{array}{ccc} K_T & \xrightarrow{\iota_T} & \mathbf{Kl}(T) \\ {\scriptstyle J_T}\uparrow & & \uparrow{\scriptstyle F_T} \\ A_f & \xrightarrow[\iota]{} & A \end{array}$$

This formulation agrees more closely with Theorem 4.1.4 but would make for slightly greater complication in our ongoing exposition.

Theorem 4.3.1. For a finitary V-monad T on A, let $F^T \dashv G^T$ be the canonical V-adjunction between the Eilenberg-Moore V-category T-$\mathbf{Alg}$ and A, and let Q^T send a T-algebra α to $T\text{-}\mathbf{Alg}(\iota_T -, \alpha)$. Then, if we allow Q^T to be replaced by a canonically isomorphic functor, the following square yields a pullback:

$$\begin{array}{ccc} T\text{-}\mathbf{Alg} & \xrightarrow{Q^T} & [L_T, V] \\ {\scriptstyle G^T}\downarrow & \lrcorner & \downarrow{\scriptstyle [J_T^{\mathrm{op}}, V]} \\ A & \xrightarrow[\tilde{\iota}]{} & [A_f^{\mathrm{op}}, V] \end{array}$$

Proof. Since $\iota_T \circ J_T = F^T \circ \iota$ and we have a V-adjunction $T\text{-}\mathbf{Alg}(F^T\iota-, -) \cong A(\iota-, G^T-)$, the square commutes up to isomorphism. As the V-functor $[J_T^{\mathrm{op}}, V]$ satisfies the isomorphism lifting property, Q^T is isomorphic to a V-functor that makes the diagram commute exactly.

Now let $a \in A$ and $M\colon L_T \to V$ satisfy $A(\iota-, a) \cong MJ_T^{\mathrm{op}}$. Using the isomorphism, the functoriality data of M yields maps

$$A(\iota m, T\iota n) \longrightarrow [A(\iota n, a), A(\iota m, a)]$$

V-natural in m and n. By V-naturality in m and by density of A_f in A, these are equivalent to maps

$$A(\iota n, a) \longrightarrow A(T\iota n, a)$$

V-natural in n, which in turn correspond to the components of a map of the form

$$\int^{n \in A_f} A(\iota n, a) \otimes T\iota n \longrightarrow a$$

with the V-naturality corresponding to the property of being a cocone. So, as $Ta = \int^{n \in A_f} A(\iota n, a) \otimes T\iota n$, the functoriality data of M yields a map $\alpha\colon Ta \to a$, cf [20, 42]. It is a T-algebra and satisfies $G^T\alpha = a$. It is routine to verify that α is the unique T-algebra such that $Q^T\alpha$ is canonically isomorphic to $T\text{-}\mathbf{Alg}(\iota_T-, \alpha)$. Conjugating with respect to isomorphisms of V-functors, one can obtain strict commutativity. Functoriality is routine. □

We remark that this theorem yields an alternative proof of the fact that the V-category of algebras for a finitary V-monad on a locally finitely presentable V-category is itself locally finitely presentable. For the fully faithfulness of Q^T shows that K_T is dense in T-**Alg**, with the objects of K_T all finitely presentable in A and hence in T-**Alg**. As T-**Alg** is also V-cocomplete, it is locally finitely presentable.

Corollary 4.3.2. The construction of T_L from an arbitrary Lawvere V-theory L and that of L from an arbitrary finitary V-monad T on A extend canonically to an equivalence of categories $\mathbf{Law}_A \simeq \mathbf{Mnd}_f(A)$. Moreover, the V-categories $\mathbf{Mod}(L)$ and T_L-$\mathbf{Alg}$ are canonically isomorphic.

Proof. By Theorem 4.3.1, $T \cong T_{L_T}$ for an arbitrary finitary V-monad T on A. Conversely, given an arbitrary Lawvere A-theory L, the Lawvere A-theory L_{T_L} is defined to be the (identity-on-objects,fully faithful) factorisation of $F^{T_L} \circ \iota \colon A_f \to T_L\text{-}\mathbf{Alg}$. By Corollary 4.2.4 and since $\mathbf{Mod}(L) \cong T_L\text{-}\mathbf{Alg}$, this factorisation agrees with L, and so L_{T_L} is isomorphic to L. The two constructions routinely extend to an equivalence of categories. □

The final line of Corollary 4.3.2 is delicate. Although there exists a canonical isomorphism as stated, it is not true that, taking V and A to be **Set**, one has an isomorphism between the usual category of models of a Lawvere theory and the category of algebras for the corresponding monad. That lack of an isomorphism is consistent with our result because our category of models is only equivalent, rather than isomorphic, to Lawvere's category.

4.4 Example: Categories with Structures

We formulate algebraic structures for terminal objects, binary products, and exponents. These structures have been presented by various way [11, 32]. First, putting $V = \mathbf{Set}$ and $A = \mathbf{Cat}_o$, we give Lawvere $\mathbf{Cat}_o$-theories for which the models are all categories with terminal objects, all categories with binary products, and all cartesian closed categories, respectively. Next, we extend some of them to those in the setting $V = A = \mathbf{Cat}$.

4.4.1 Categories with a Terminal Object

First, we define properties T_1 through T_4 for categories and show rule-based presentations of the properties. We prove that a category C satisfies them if and only if C has a terminal object. Next, we construct a Lawvere $\mathbf{Cat}_o$-theory and show the rule-based presentation of the Lawvere $\mathbf{Cat}_o$-theory. Last, we need to prove that C has a terminal object if and only if C is a model of the Lawvere $\mathbf{Cat}_o$-theory. By comparing both rule-based presentations, we can prove that syntactically.

In other sections, we also use rule-based presentations for other examples. Shapes of judgements depend on the base category A of the intended Lawvere A-theory. Since A is $\mathbf{Cat}_o$ in this section, we use judgements for objects, arrows, and equations between arrows as follows.

Let C be an arbitrary category. A judgement $x\colon \mathbf{obj}$ represents that x is an object of C. A judgement $f\colon x \to y$ represents that f is an arrow from x to y in C. A judgement $f = g\colon x \to y$ represents that $f, g\colon x \to y$ are arrows of C and that f is equal to g. We write $\mathbf{Id}$ for the identity arrow.

A category C satisfies T_1 if there exists an object 1 in C.

$$\frac{}{1\colon \mathbf{obj}}\, T_1$$

A category C satisfies T_2 if for object x in C, there exists an arrow $!_x\colon x \to 1$ in C.

$$\frac{x\colon \mathbf{obj}}{!_x\colon x \to 1}\, T_2$$

A category C satisfies T_3 if the arrow $!_1$ is equal to the identity arrow on the object 1 in C.

$$\frac{}{!_1 = \mathbf{Id}\colon 1 \to 1}\, T_3$$

A category C satisfies T_4 if for arrow $f\colon x \to y$ in C, the composition of $!_y$ and f is equal to the arrow $!_x\colon x \to 1$ in C.

$$\frac{f\colon x \to y}{!_y \circ f = !_x\colon x \to 1}\, T_4$$

Theorem 4.4.1 (See [11])**.** A category C satisfies T_1 through T_4 if and only if C has a terminal object.

Our goal is to define the Lawvere $\mathbf{Cat}_o$-theory for which models are all categories with terminal objects. We can check $\mathbf{Cat}_o$ is a locally finitely presentable category. For later use, we define some finitely presentable objects in $\mathbf{Cat}_o$.

- 0 is the empty category (no objects, no arrows).

- 1 is the category with one object and one (identity) arrow.
- **2** (or $\rightarrow$) is the category freely generated from the following graph.

$$\cdot \longrightarrow \cdot$$

- **3** is the category freely generated from the following graph.

$$\cdot \longrightarrow \cdot \longrightarrow \cdot$$

We define the Lawvere $\mathbf{Cat}_o$-theory that corresponds to the rule-based presentation for categories with terminal objects. For each rule, we add one new arrow to the category $(\mathbf{Cat}_o)_f^{\mathrm{op}}$. The shapes of the premise and the consequence parts of a rule determine the domain and codomain objects (which are categories). Let $\mathbf{Ct0} = (L, J)$ be the freely generated Lawvere $\mathbf{Cat}_o$-theory from $(\mathbf{Cat}_o)_f^{\mathrm{op}}$ by adding the new arrows called as follows.

$$\begin{aligned} \mathbf{T}_1 &\colon\ 0 \rightarrow 1 \\ \mathbf{T}_2 &\colon\ 1 \rightarrow \mathbf{2} \\ \mathbf{T}_3 &\colon\ 0 \rightarrow \mathbf{2} \\ \mathbf{T}_4 &\colon\ \mathbf{2} \rightarrow \mathbf{3} \end{aligned}$$

Let (C, S) be a model of $\mathbf{Ct0}$. By the definition of models, C is an object of $\mathbf{Cat}_o$ and S is a functor $S\colon \mathbf{Ct0} \rightarrow \mathbf{Set}$. The functor S sends the arrow $\mathbf{T}_2$ to a function $S\mathbf{T}_2\colon S1 \rightarrow S\mathbf{2}$. Since the object part of J is the identity, all objects x in $(\mathbf{Cat}_o)_f^{\mathrm{op}}$ satisfies $Sx = SJx$. By the definition of models, x satisfies $SJx = \mathbf{Cat}_o(\iota x, C) = \mathbf{Cat}_o(x, C)$. Therefore, the function $S\mathbf{T}_2$ sends an element of $\mathbf{Cat}_o(1, C)$ to an element of $\mathbf{Cat}_o(\mathbf{2}, C)$. An element of $\mathbf{Cat}_o(1, C)$ corresponds to an object in C. An element of $\mathbf{Cat}_o(\mathbf{2}, C)$ corresponds to an arrow in C. Therefore, $S\mathbf{T}_2$ represents that for object x in C, there exist an arrow $!_x\colon y \rightarrow z$ in C.

$$\frac{x\colon \mathbf{obj}}{!_x\colon y \rightarrow z}\ \mathbf{T}_2$$

However, this property of C is not logically equivalent to the property T_2. Remember that C satisfies the property T_2 if for object x in C, there exists an arrow

$!_x\colon x \to 1$. When $S\mathbf{T}_2$ sends x to $!_x\colon y \to z$, however, y is not always equal to x.

$$\frac{x\colon \mathbf{obj}}{!_x\colon x \to 1}\, T_2$$

Therefore, we give certain equations among arrows of the Lawvere $\mathbf{Cat}_o$-theory $\mathbf{Ct0}$. Arrows of $\mathbf{Ct0}$ consist of the above new arrows $\mathbf{T}_1$ through $\mathbf{T}_4$ and arrows of $(\mathbf{Cat}_o)_f^{\mathrm{op}}$. Equations among arrows of $\mathbf{Ct0}$ are preserved by S of all model (C, S). Therefore, they represent conditions of functions $S\mathbf{T}_1$ through $S\mathbf{T}_4$. We precisely define them as follows. First, we define arrows $\mathbf{G}_1$ through $\mathbf{G}_7$ in $\mathbf{Ct0}$ by showing functions $S\mathbf{G}_1$ through $S\mathbf{G}_7$ for a model (C, S) of $\mathbf{Ct0}$. By the definition of models, all arrow G in $(\mathbf{Cat}_o)_f^{\mathrm{op}}$ satisfies $SJG = \mathbf{Cat}_o(\iota G, C)$.

There exists $\mathbf{G}_1\colon 1 \to 0$ such that for object x in C, the function $S\mathbf{G}_1$ returns the unique object of $\mathbf{Cat}_o(0, C)$. The arrow $\mathbf{G}_1$ corresponds to the unique arrow from 0 to 1 in $(\mathbf{Cat}_o)_f$.

$$\frac{x\colon \mathbf{obj}}{\cdot}\, \mathbf{G}_1$$

There exists $\mathbf{G}_2\colon 1 \to \mathbf{2}$ such that for object x in C, the function $S\mathbf{G}_2$ returns the identity arrow $\mathbf{Id}\colon x \to x$. The arrow $\mathbf{G}_2$ corresponds to the unique arrow from $\mathbf{2}$ to 1 in $(\mathbf{Cat}_o)_f$.

$$\frac{x\colon \mathbf{obj}}{\mathbf{Id}\colon x \to x}\, \mathbf{G}_2$$

There exists $\mathbf{G}_3\colon \mathbf{2} \to 1$ such that for arrow $f\colon x \to y$ in C, the function $S\mathbf{G}_3$ returns x. The arrow $\mathbf{G}_3$ corresponds to the arrow in $(\mathbf{Cat}_o)_f$ which sends the unique object of 1 to the domain of the unique non-identity arrow in $\mathbf{2}$. There exists $\mathbf{G}_4\colon \mathbf{2} \to 1$ such that for arrow $f\colon x \to y$ in C, the function $S\mathbf{G}_4$ returns y. The arrow $\mathbf{G}_4$ corresponds to the arrow in $(\mathbf{Cat}_o)_f$ which sends the unique object of 1 to the codomain of the unique non-identity arrow in $\mathbf{2}$.

$$\frac{f\colon x \to y}{x\colon \mathbf{obj}}\, \mathbf{G}_3 \quad \frac{f\colon x \to y}{y\colon \mathbf{obj}}\, \mathbf{G}_4$$

There exist $\mathbf{G}_5, \mathbf{G}_6, \mathbf{G}_7\colon \mathbf{3} \to \mathbf{2}$ such that for arrow $f\colon x \to y$ and arrow $g\colon y \to z$ in C, the functions $S\mathbf{G}_5, S\mathbf{G}_6, S\mathbf{G}_7$ return f, g, and $g \circ f$, respectively.

$$\frac{f\colon x \to y \quad g\colon y \to z}{f\colon x \to y}\, \mathbf{G}_5 \quad \frac{f\colon x \to y \quad g\colon y \to z}{g\colon y \to z}\, \mathbf{G}_6 \quad \frac{f\colon x \to y \quad g\colon y \to z}{g \circ f\colon x \to z}\, \mathbf{G}_7$$

Let **Ct** be the freely generated Lawvere $\mathbf{Cat}_o$-theory from **Ct0** subject to the following equations.

$$\begin{aligned}
\mathbf{G}_4 \circ \mathbf{T}_2 &= \mathbf{T}_1 \circ \mathbf{G}_1 \\
\mathbf{G}_3 \circ \mathbf{T}_2 &= \mathbf{Id} \\
\mathbf{T}_3 &= \mathbf{T}_2 \circ \mathbf{T}_1 \\
\mathbf{T}_3 &= \mathbf{G}_2 \circ \mathbf{T}_1 \\
\mathbf{G}_5 \circ \mathbf{T}_4 &= \mathbf{Id} \\
\mathbf{G}_6 \circ \mathbf{T}_4 &= \mathbf{T}_2 \circ \mathbf{G}_4 \\
\mathbf{G}_7 \circ \mathbf{T}_4 &= \mathbf{T}_2 \circ \mathbf{G}_3
\end{aligned}$$

Theorem 4.4.2. There exists a Lawvere $\mathbf{Cat}_o$-theory **Ct** for which the models are all categories with terminal object.

We can also reformulate by a single operation for plural operations that have a common domain. However, since it is difficult to understand the correspondence between the reformulated Lawvere $\mathbf{Cat}_o$-theory and the properties, we do not so.

4.4.2 Categories with Binary Products

Similarly to Section 4.4.1, we define properties B_1 through B_8 for categories. Let C be an arbitrary category. A category C satisfies B_1 if for object x and y in C, there exists an object $x \times y$ in C.

$$\frac{x\colon \mathbf{obj} \quad y\colon \mathbf{obj}}{x \times y\colon \mathbf{obj}}\ B_1$$

A category C satisfies B_2 if for object x and y in C, there exists an arrow $\lambda_{x,y}\colon x \times y \to x$ in C.

$$\frac{x\colon \mathbf{obj} \quad y\colon \mathbf{obj}}{\lambda_{x,y}\colon x \times y \to x}\ B_2$$

A category C satisfies B_3 if for object x and y in C, there exists an arrow $\rho_{x,y}\colon x \times y \to y$ in C.

$$\frac{x\colon \mathbf{obj} \quad y\colon \mathbf{obj}}{\rho_{x,y}\colon x \times y \to y}\ B_3$$

A category C satisfies B_4 if for arrow $f\colon x \to y$ and $g\colon x \to z$ in C, there exists an arrow $\langle f, g\rangle\colon x \to y \times z$ in C.

$$\frac{f\colon x \to y \quad g\colon x \to z}{\langle f, g\rangle\colon x \to y \times z}\ B_4$$

A category C satisfies B_5 if for object x and y in C, the arrow $\langle \lambda_{x,y}, \rho_{x,y} \rangle$ is equal to the identity arrow on the object $x \times y$ in C.

$$\frac{x\colon \mathbf{obj} \quad y\colon \mathbf{obj}}{\langle \lambda_{x,y}, \rho_{x,y} \rangle = \mathbf{Id}\colon x \times y \to x \times y}\ B_5$$

A category C satisfies B_6 if for arrow $f\colon x \to y$ and $g\colon x \to z$ in C, the arrow $\lambda_{y,z} \circ \langle f, g \rangle$ is equal to the arrow f.

$$\frac{f\colon x \to y \quad g\colon x \to z}{\lambda_{y,z} \circ \langle f, g \rangle = f\colon x \to y}\ B_6$$

A category C satisfies B_7 if for arrow $f\colon x \to y$ and $g\colon x \to z$ in C, the arrow $\rho_{y,z} \circ \langle f, g \rangle$ is equal to the arrow g.

$$\frac{f\colon x \to y \quad g\colon x \to z}{\rho_{y,z} \circ \langle f, g \rangle = g\colon x \to z}\ B_7$$

A category C satisfies B_8 if for arrow $f\colon x \to y$, $g\colon x \to z$, and $h\colon w \to x$ in C, the arrow $\langle f, g \rangle \circ h$ is equal to the arrow $\langle f \circ h, g \circ h \rangle$.

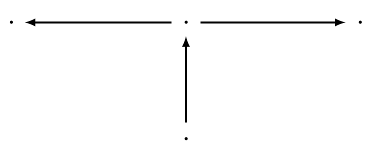
$$\frac{h\colon w \to x \quad f\colon x \to y \quad g\colon x \to z}{\langle f, g \rangle \circ h = \langle f \circ h, g \circ h \rangle\colon w \to y \times z}\ B_8$$

Theorem 4.4.3. A category C satisfies B_1 through B_8 if and only if C has binary products.

For later use, we define some finitely presentable objects in $\mathbf{Cat}_o$.

- 2 is the category with two objects.
- $\mathbf{A_1}$ is the category freely generated from the following graph.

- $\mathbf{A_2}$ is the category freely generated from the following graph.



Let **Cb0** be the freely generated Lawvere $\mathbf{Cat}_o$-theory from $(\mathbf{Cat}_o)_f^{\mathrm{op}}$ by adding the new arrows called as follows.

$$\begin{array}{ll} \mathbf{B}_1\colon\ 2 \to 1 & \mathbf{B}_5\colon\ 2 \to \mathbf{A_1} \\ \mathbf{B}_2\colon\ 2 \to \mathbf{2} & \mathbf{B}_6\colon\ \mathbf{A_1} \to \mathbf{3} \\ \mathbf{B}_3\colon\ 2 \to \mathbf{2} & \mathbf{B}_7\colon\ \mathbf{A_1} \to \mathbf{3} \\ \mathbf{B}_4\colon\ \mathbf{A_1} \to \mathbf{2} & \mathbf{B}_8\colon\ \mathbf{A_2} \to \mathbf{3} \end{array}$$

Next, we give certain equations among arrows of the Lawvere $\mathbf{Cat}_o$-theory **Cb0**. Let (C, S) be a model of **Cb0**.

There exists $\mathbf{G}_8\colon\ 2 \to 1$ such that for objects x and y in C, the function $S\mathbf{G}_8$ returns x.

$$\frac{x\colon\ \mathbf{obj} \quad y\colon\ \mathbf{obj}}{x\colon\ \mathbf{obj}}\ \mathbf{G}_8$$

There exists $\mathbf{G}_9\colon\ 2 \to 1$ such that for objects x and y in C, the function $S\mathbf{G}_9$ returns y.

$$\frac{x\colon\ \mathbf{obj} \quad y\colon\ \mathbf{obj}}{y\colon\ \mathbf{obj}}\ \mathbf{G}_9$$

There exists $\mathbf{G}_{10}\colon\ \mathbf{A_1} \to 1$ such that for arrow $f\colon\ x \to y$ and arrow $g\colon\ x \to z$ in C, the function $S\mathbf{G}_{10}$ returns x.

$$\frac{f\colon\ x \to y \quad g\colon\ x \to z}{x\colon\ \mathbf{obj}}\ \mathbf{G}_{10}$$

There exists $\mathbf{G}_{11}\colon\ \mathbf{A_1} \to 2$ such that for arrow $f\colon\ x \to y$ and arrow $g\colon\ x \to z$ in C, the function $S\mathbf{G}_{11}$ returns y and z.

$$\frac{f\colon\ x \to y \quad g\colon\ x \to z}{y, z\colon\ \mathbf{obj}}\ \mathbf{G}_{11}$$

There exists $\mathbf{G}_{12}\colon\ \mathbf{A_1} \to \mathbf{2}$ such that for arrow $f\colon\ x \to y$ and arrow $g\colon\ x \to z$ in C, the function $S\mathbf{G}_{12}$ returns the identity arrow on x.

$$\frac{f\colon\ x \to y \quad g\colon\ x \to z}{\mathbf{Id}\colon\ x \to x}\ \mathbf{G}_{12}$$

There exists $\mathbf{G}_{13}\colon\ \mathbf{A_1} \to \mathbf{2}$ such that for arrow $f\colon\ x \to y$ and arrow $g\colon\ x \to z$ in C, the function $S\mathbf{G}_{13}$ returns the arrow f.

$$\frac{f\colon\ x \to y \quad g\colon\ x \to z}{f\colon\ x \to y}\ \mathbf{G}_{13}$$

There exists $\mathbf{G}_{14}\colon \mathbf{A_1} \to \mathbf{2}$ such that for arrow $f\colon x \to y$ and arrow $g\colon x \to z$ in C, the function $S\mathbf{G}_{14}$ returns the arrow g.

$$\frac{f\colon x \to y \quad g\colon x \to z}{g\colon x \to z}\ \mathbf{G}_{14}$$

There exists $\mathbf{G}_{15}\colon \mathbf{A_2} \to \mathbf{2}$ such that for arrow $f\colon x \to y$, $g\colon x \to z$, and $h\colon w \to x$ in C, the function $S\mathbf{G}_{15}$ returns the arrow h.

$$\frac{h\colon w \to x \quad f\colon x \to y \quad g\colon x \to z}{h\colon w \to x}\ \mathbf{G}_{15}$$

There exists $\mathbf{G}_{16}\colon \mathbf{A_2} \to \mathbf{A_1}$ such that for arrow $f\colon x \to y$, $g\colon x \to z$, and $h\colon w \to x$ in C, the function $S\mathbf{G}_{16}$ returns $f \circ h$ and $g \circ h$.

$$\frac{h\colon w \to x \quad f\colon x \to y \quad g\colon x \to z}{f \circ h\colon w \to y \quad g \circ h\colon w \to z}\ \mathbf{G}_{16}$$

There exists $\mathbf{G}_{17}\colon \mathbf{A_2} \to \mathbf{A_1}$ such that for arrow $f\colon x \to y$, $g\colon x \to z$, and $h\colon w \to x$ in C, the function $S\mathbf{G}_{17}$ returns f and g.

$$\frac{h\colon w \to x \quad f\colon x \to y \quad g\colon x \to z}{f\colon x \to y \quad g\colon x \to z}\ \mathbf{G}_{17}$$

Let **Cb** be the freely generated Lawvere $\mathbf{Cat}_o$-theory from **Cb0** subject to the following equations.

$$\begin{array}{rclrcl}
\mathbf{G}_3 \circ \mathbf{B}_2 &=& \mathbf{B}_1 & \mathbf{G}_5 \circ \mathbf{B}_6 &=& \mathbf{B}_4 \\
\mathbf{G}_4 \circ \mathbf{B}_2 &=& \mathbf{G}_8 & \mathbf{G}_6 \circ \mathbf{B}_6 &=& \mathbf{B}_2 \circ \mathbf{G}_{11} \\
\mathbf{G}_3 \circ \mathbf{B}_3 &=& \mathbf{B}_1 & \mathbf{G}_7 \circ \mathbf{B}_6 &=& \mathbf{G}_{13} \\
\mathbf{G}_4 \circ \mathbf{B}_3 &=& \mathbf{G}_9 & \mathbf{G}_5 \circ \mathbf{B}_7 &=& \mathbf{B}_4 \\
\mathbf{G}_3 \circ \mathbf{B}_4 &=& \mathbf{G}_{10} & \mathbf{G}_6 \circ \mathbf{B}_7 &=& \mathbf{B}_3 \circ \mathbf{G}_{11} \\
\mathbf{G}_4 \circ \mathbf{B}_4 &=& \mathbf{B}_1 \circ \mathbf{G}_{11} & \mathbf{G}_7 \circ \mathbf{B}_7 &=& \mathbf{G}_{14} \\
\mathbf{G}_{12} \circ \mathbf{B}_5 &=& \mathbf{G}_2 \circ \mathbf{B}_1 & \mathbf{G}_5 \circ \mathbf{B}_8 &=& \mathbf{G}_{15} \\
\mathbf{G}_{13} \circ \mathbf{B}_5 &=& \mathbf{B}_2 & \mathbf{G}_6 \circ \mathbf{B}_8 &=& \mathbf{B}_4 \circ \mathbf{G}_{17} \\
\mathbf{G}_{14} \circ \mathbf{B}_5 &=& \mathbf{B}_3 & \mathbf{G}_7 \circ \mathbf{B}_8 &=& \mathbf{B}_4 \circ \mathbf{G}_{16}
\end{array}$$

Theorem 4.4.4. There exists a Lawvere $\mathbf{Cat}_o$-theory **Cb** for which the models are all categories with binary products.

4.4.3 Cartesian Closed Categories

Similarly to Section 4.4.1, we define properties E_1 through E_{10} for categories. Let C be an arbitrary category with finite products. We write $x \times f$ for the following arrow.

$$\frac{x\colon \mathbf{obj} \quad f\colon y \to z}{\langle \lambda_{x,y}, f \circ \rho_{x,y} \rangle\colon x \times y \to x \times z}$$

A category C satisfies E_1 if for object x and y in C, there exists an object $[x, y]$ in C.

$$\frac{x\colon \mathbf{obj} \quad y\colon \mathbf{obj}}{[x, y]\colon \mathbf{obj}}\ E_1$$

A category C satisfies E_2 if for object x and arrow $f\colon y \to z$ in C, there exists an arrow $[x, f]\colon [x, y] \to [x, z]$ in C.

$$\frac{x\colon \mathbf{obj} \quad f\colon y \to z}{[x, f]\colon [x, y] \to [x, z]}\ E_2$$

A category C satisfies E_3 if for object x and y in C, the arrow $[x, \mathbf{Id}]$ is equal to the identity arrow on $[x, y]$.

$$\frac{x\colon \mathbf{obj} \quad y\colon \mathbf{obj}}{[x, \mathbf{Id}] = \mathbf{Id}\colon [x, y] \to [x, y]}\ E_3$$

A category C satisfies E_4 if for object w, arrow $f\colon x \to y$, and $g\colon y \to z$ in C, the arrow $[w, g \circ f]$ is equal to the arrow $[w, g] \circ [w, f]$.

$$\frac{w\colon \mathbf{obj} \quad f\colon x \to y \quad g\colon y \to z}{[w, g \circ f] = [w, g] \circ [w, f]\colon [w, x] \to [w, z]}\ E_4$$

A category C satisfies E_5 if for object x and y in C, there exists an arrow $\eta_x(y)\colon y \to [x, x \times y]$ in C.

$$\frac{x\colon \mathbf{obj} \quad y\colon \mathbf{obj}}{\eta_x(y)\colon y \to [x, x \times y]}\ E_5$$

A category C satisfies E_6 if for object x and y in C, there exists an arrow $\epsilon_x(y)\colon x \times [x, y] \to y$ in C.

$$\frac{x\colon \mathbf{obj} \quad y\colon \mathbf{obj}}{\epsilon_x(y)\colon x \times [x, y] \to y}\ E_6$$

A category C satisfies E_7 if for object x and arrow $f\colon y \to z$, the arrow $\eta_x(z) \circ f$ is equal to the arrow $[x, x \times f] \circ \eta_x(y)$.

$$\frac{x\colon \mathbf{obj} \quad f\colon y \to z}{\eta_x(z) \circ f = [x, x \times f] \circ \eta_x(y)\colon y \to [x, x \times z]}\ E_7$$

A category C satisfies E_8 if for object x and arrow $f\colon y \to z$, the arrow $\epsilon_x(z) \circ x \times [x, f]$ is equal to the arrow $f \circ \epsilon_x(y)$.

$$\frac{x\colon \mathbf{obj} \quad f\colon y \to z}{\epsilon_x(z) \circ x \times [x, f] = f \circ \epsilon_x(y)\colon x \times [x, y] \to z}\ E_8$$

A category C satisfies E_9 if for object x and y, the arrow $[x, \epsilon_x(y)] \circ \eta_x([x, y])$ is equal to the identity arrow on $[x, y]$.

$$\frac{x\colon \mathbf{obj} \quad y\colon \mathbf{obj}}{[x, \epsilon_x(y)] \circ \eta_x([x, y]) = \mathbf{Id}\colon [x, y] \to [x, y]}\ E_9$$

A category C satisfies E_{10} if for object x and y, the arrow $\epsilon_x(x \times y) \circ (x \times \eta_x(y))$ is equal to the identity arrow on $x \times y$.

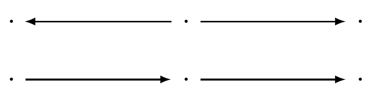
$$\frac{x\colon \mathbf{obj} \quad y\colon \mathbf{obj}}{\epsilon_x(x \times y) \circ (x \times \eta_x(y)) = \mathbf{Id}\colon x \times y \to x \times y}\ E_{10}$$

Theorem 4.4.5. A category C with finite products satisfies E_1 through E_8 if and only if C is a Cartesian closed category.

For later use, we define some finitely presentable objects in $\mathbf{Cat}_o$.

- $1 + \mathbf{2}$ is the category freely generated from the following graph.

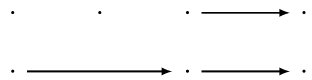
- $\mathbf{A_1} + \mathbf{3}$ is the category freely generated from the following graph.



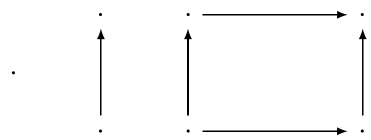
- $1 + \mathbf{3}$ is the category freely generated from the following graph.

- $2 + \mathbf{2}$ is the category freely generated from the following graph.

- $2 + \mathbf{2} + \mathbf{3}$ is the category freely generated from the following graph.



- **Natrl** is the category freely generated from the following graph subject to the commutative diagram.



Let **Ccc0** be the freely generated Lawvere $\mathbf{Cat}_o$-theory from $(\mathbf{Cat}_o)_f^{\mathrm{op}}$ by adding $\mathbf{T}_1$ through $\mathbf{T}_4$ of **Ct0**, $\mathbf{B}_1$ through $\mathbf{B}_8$ of **Cb0**, and the new arrows called as follows.

$\mathbf{B}_9 \colon 1 + \mathbf{2} \to \mathbf{A_1} + \mathbf{3}$
$\mathbf{E}_1 \colon 2 \to 1$
$\mathbf{E}_2 \colon 1 + \mathbf{2} \to \mathbf{2}$
$\mathbf{E}_3 \colon 2 \to \mathbf{2}$
$\mathbf{E}_4 \colon 1 + \mathbf{3} \to \mathbf{3}$
$\mathbf{E}_5 \colon 2 \to 2 + \mathbf{2}$
$\mathbf{E}_6 \colon 2 \to 2 + \mathbf{2}$
$\mathbf{E}_7 \colon 1 + \mathbf{2} \to \mathbf{Natrl}$
$\mathbf{E}_8 \colon 1 + \mathbf{2} \to \mathbf{Natrl}$
$\mathbf{E}_9 \colon 2 \to 2 + \mathbf{2} + \mathbf{3}$
$\mathbf{E}_{10} \colon 2 \to 2 + \mathbf{2} + \mathbf{3}$

Next, we give certain equations among arrows of the Lawvere $\mathbf{Cat}_o$-theory **Ccc0**. Let (C, S) be a model of **Ccc0**.

There exists $\mathbf{G}_{18} \colon 1 + \mathbf{2} \to 2$ such that for object x and arrow $f \colon y \to z$ in C, the function $S\mathbf{G}_{18}$ returns x and y.

$$\frac{x \colon \mathbf{obj} \quad f \colon y \to z}{x, y \colon \mathbf{obj}} \,\mathbf{G}_{18}$$

There exists $\mathbf{G}_{19} \colon 1 + \mathbf{2} \to 2$ such that for object x and arrow $f \colon y \to z$ in C, the function $S\mathbf{G}_{19}$ returns x and z.

$$\frac{x \colon \mathbf{obj} \quad f \colon y \to z}{x, z \colon \mathbf{obj}} \,\mathbf{G}_{19}$$

There exists $\mathbf{G}_{20}\colon 2 \to 1 + \mathbf{2}$ such that for objects x and y in C, the function $S\mathbf{G}_{20}$ returns x and the identity arrow on y.

$$\frac{x\colon \mathbf{obj} \quad y\colon \mathbf{obj}}{x\colon \mathbf{obj} \quad \mathbf{Id}\colon y \to y}\ \mathbf{G}_{20}$$

There exists $\mathbf{G}_{21}\colon 1 + \mathbf{3} \to 1 + \mathbf{2}$ such that for object w, arrow $f\colon x \to y$, and $g\colon y \to z$ in C, the function $S\mathbf{G}_{21}$ returns x and $f\colon x \to y$.

$$\frac{w\colon \mathbf{obj} \quad f\colon x \to y \quad g\colon y \to z}{w\colon \mathbf{obj} \quad f\colon x \to y}\ \mathbf{G}_{21}$$

There exists $\mathbf{G}_{22}\colon 1 + \mathbf{3} \to 1 + \mathbf{2}$ such that for object w, arrow $f\colon x \to y$, and $g\colon y \to z$ in C, the function $S\mathbf{G}_{22}$ returns x and $g\colon y \to z$.

$$\frac{w\colon \mathbf{obj} \quad f\colon x \to y \quad g\colon y \to z}{w\colon \mathbf{obj} \quad g\colon y \to z}\ \mathbf{G}_{22}$$

There exists $\mathbf{G}_{23}\colon 1 + \mathbf{3} \to 1 + \mathbf{2}$ such that for object w, arrow $f\colon x \to y$, and $g\colon y \to z$ in C, the function $S\mathbf{G}_{23}$ returns x and $g \circ f\colon x \to z$.

$$\frac{w\colon \mathbf{obj} \quad f\colon x \to y \quad g\colon y \to z}{w\colon \mathbf{obj} \quad g \circ f\colon x \to z}\ \mathbf{G}_{23}$$

There exists $\mathbf{G}_{24}\colon 2 + \mathbf{2} \to 2$ such that for object v, w, and arrow $f\colon x \to y$ in C, the function $S\mathbf{G}_{24}$ returns v and w.

$$\frac{v\colon \mathbf{obj} \quad w\colon \mathbf{obj} \quad f\colon x \to y}{v, w\colon \mathbf{obj}}\ \mathbf{G}_{24}$$

There exists $\mathbf{G}_{25}\colon 2 + \mathbf{2} \to \mathbf{2}$ such that for object v, w, and arrow $f\colon x \to y$ in C, the function $S\mathbf{G}_{25}$ returns f.

$$\frac{v\colon \mathbf{obj} \quad w\colon \mathbf{obj} \quad f\colon x \to y}{f\colon x \to y}\ \mathbf{G}_{25}$$

There exists $\mathbf{G}_{26}\colon \mathbf{Natrl} \to 1 + \mathbf{2}$ such that for object x, arrows $f\colon y \to z$, $g\colon a \to b$, $h\colon b \to c$, $k\colon a \to c$, and $l\colon c \to d$ in C such that $h \circ g = l \circ k$, the function $S\mathbf{G}_{26}$ returns x and f.

$$\frac{x\colon \mathbf{obj} \quad f\colon y \to z \quad g\colon a \to b \quad h\colon b \to c \quad k\colon a \to c \quad l\colon c \to d \quad h \circ g = l \circ k}{x\colon \mathbf{obj} \quad f\colon y \to z}\ \mathbf{G}_{26}$$

There exists $\mathbf{G}_{27}\colon\ \mathbf{Natrl} \to \mathbf{2}$ such that for object x, arrows $f\colon\ y \to z$, $g\colon\ a \to b$, $h\colon\ b \to c$, $k\colon\ a \to c$, and $l\colon\ c \to d$ in C such that $h \circ g = l \circ k$, the function $S\mathbf{G}_{27}$ returns g.

$$\frac{x\colon\ \mathbf{obj} \quad f\colon\ y \to z \quad g\colon\ a \to b \quad h\colon\ b \to c \quad k\colon\ a \to c \quad l\colon\ c \to d \quad h \circ g = l \circ k}{g\colon\ a \to b}\ \mathbf{G}_{27}$$

There exists $\mathbf{G}_{28}\colon\ \mathbf{Natrl} \to \mathbf{2}$ such that for object x, arrows $f\colon\ y \to z$, $g\colon\ a \to b$, $h\colon\ b \to c$, $k\colon\ a \to c$, and $l\colon\ c \to d$ in C such that $h \circ g = l \circ k$, the function $S\mathbf{G}_{28}$ returns h.

$$\frac{x\colon\ \mathbf{obj} \quad f\colon\ y \to z \quad g\colon\ a \to b \quad h\colon\ b \to c \quad k\colon\ a \to c \quad l\colon\ c \to d \quad h \circ g = l \circ k}{h\colon\ b \to c}\ \mathbf{G}_{28}$$

There exists $\mathbf{G}_{29}\colon\ \mathbf{Natrl} \to \mathbf{2}$ such that for object x, arrows $f\colon\ y \to z$, $g\colon\ a \to b$, $h\colon\ b \to c$, $k\colon\ a \to c$, and $l\colon\ c \to d$ in C such that $h \circ g = l \circ k$, the function $S\mathbf{G}_{29}$ returns k.

$$\frac{x\colon\ \mathbf{obj} \quad f\colon\ y \to z \quad g\colon\ a \to b \quad h\colon\ b \to c \quad k\colon\ a \to c \quad l\colon\ c \to d \quad h \circ g = l \circ k}{k\colon\ a \to c}\ \mathbf{G}_{29}$$

There exists $\mathbf{G}_{30}\colon\ \mathbf{Natrl} \to \mathbf{2}$ such that for object x, arrows $f\colon\ y \to z$, $g\colon\ a \to b$, $h\colon\ b \to c$, $k\colon\ a \to c$, and $l\colon\ c \to d$ in C such that $h \circ g = l \circ k$, the function $S\mathbf{G}_{30}$ returns l.

$$\frac{x\colon\ \mathbf{obj} \quad f\colon\ y \to z \quad g\colon\ a \to b \quad h\colon\ b \to c \quad k\colon\ a \to c \quad l\colon\ c \to d \quad h \circ g = l \circ k}{l\colon\ c \to d}\ \mathbf{G}_{30}$$

There exists $\mathbf{G}_{31}\colon\ 1 + \mathbf{2} \to 1$ such that for object x and arrow $f\colon\ y \to z$ in C, the function $S\mathbf{G}_{31}$ returns x.

$$\frac{x\colon\ \mathbf{obj} \quad f\colon\ y \to z}{x\colon\ \mathbf{obj}}\ \mathbf{G}_{31}$$

There exists $\mathbf{G}_{32}\colon\ 1 + \mathbf{2} \to \mathbf{2}$ such that for object x and arrow $f\colon\ y \to z$ in C, the function $S\mathbf{G}_{32}$ returns f.

$$\frac{x\colon\ \mathbf{obj} \quad f\colon\ y \to z}{f\colon\ y \to z}\ \mathbf{G}_{32}$$

There exists $\mathbf{G}_{33}\colon\ 2+\mathbf{2}+\mathbf{3} \to 1+\mathbf{2}$ such that for objects x and w, arrows $f\colon\ y \to z$, $g\colon\ a \to b$, and $h\colon\ b \to c$, the function $S\mathbf{G}_{33}$ returns x and f.

$$\frac{x\colon\ \mathbf{obj} \quad w\colon\ \mathbf{obj} \quad f\colon\ y \to z \quad g\colon\ a \to b \quad h\colon\ b \to c}{x\colon\ \mathbf{obj} \quad f\colon\ y \to z}\ \mathbf{G}_{33}$$

There exists $\mathbf{G}_{34}\colon 2+\mathbf{2}+\mathbf{3}\to 2$ such that for objects x and w, arrows $f\colon y\to z$, $g\colon a\to b$, and $h\colon b\to c$, the function $S\mathbf{G}_{34}$ returns x and w.

$$\frac{x\colon \mathbf{obj}\quad w\colon \mathbf{obj}\quad f\colon y\to z\quad g\colon a\to b\quad h\colon b\to c}{x,w\colon \mathbf{obj}}\ \mathbf{G}_{34}$$

There exists $\mathbf{G}_{35}\colon 2+\mathbf{2}+\mathbf{3}\to \mathbf{3}$ such that for objects x and w, arrows $f\colon y\to z$, $g\colon a\to b$, and $h\colon b\to c$, the function $S\mathbf{G}_{35}$ returns g and h.

$$\frac{x\colon \mathbf{obj}\quad w\colon \mathbf{obj}\quad f\colon y\to z\quad g\colon a\to b\quad h\colon b\to c}{g\colon a\to b\quad h\colon b\to c}\ \mathbf{G}_{35}$$

There exists $\mathbf{G}_{36}\colon \mathbf{A_1}+\mathbf{3}\to \mathbf{A_1}$ such that for arrows $f\colon x\to y$, $g\colon x\to z$, $h\colon a\to b$, and $k\colon b\to c$, the function $S\mathbf{G}_{36}$ returns f and g.

$$\frac{f\colon x\to y\quad g\colon x\to z\quad h\colon a\to b\quad k\colon b\to c}{f\colon x\to y\quad g\colon x\to z}\ \mathbf{G}_{36}$$

There exists $\mathbf{G}_{37}\colon \mathbf{A_1}+\mathbf{3}\to \mathbf{3}$ such that for arrows $f\colon x\to y$, $g\colon x\to z$, $h\colon a\to b$, and $k\colon b\to c$, the function $S\mathbf{G}_{37}$ returns h and k.

$$\frac{f\colon x\to y\quad g\colon x\to z\quad h\colon a\to b\quad k\colon b\to c}{h\colon a\to b\quad k\colon b\to c}\ \mathbf{G}_{37}$$

Let **Ccc** be the freely generated Lawvere $\mathbf{Cat}_o$-theory from **Ccc0** subject to the following equations and the same equations as **Ct** and **Cb**.

$$\begin{aligned}
\mathbf{G}_{13}\circ\mathbf{G}_{36}\circ\mathbf{B}_9 &= \mathbf{B}_2\circ\mathbf{G}_{18}\\
\mathbf{G}_{5}\circ\mathbf{G}_{37}\circ\mathbf{B}_9 &= \mathbf{B}_3\circ\mathbf{G}_{18}\\
\mathbf{G}_{6}\circ\mathbf{G}_{37}\circ\mathbf{B}_9 &= \mathbf{G}_{32}\\
\mathbf{G}_{7}\circ\mathbf{G}_{37}\circ\mathbf{B}_9 &= \mathbf{G}_{14}\circ\mathbf{G}_{36}\circ\mathbf{B}_9
\end{aligned}
\qquad
\begin{aligned}
\mathbf{G}_3\circ\mathbf{E}_2 &= \mathbf{E}_1\circ\mathbf{G}_{18}\\
\mathbf{G}_4\circ\mathbf{E}_2 &= \mathbf{E}_1\circ\mathbf{G}_{19}\\
\mathbf{E}_3 &= \mathbf{E}_2\circ\mathbf{G}_{20}\\
\mathbf{E}_3 &= \mathbf{G}_2\circ\mathbf{E}_1\\
\mathbf{G}_5\circ\mathbf{E}_4 &= \mathbf{E}_2\circ\mathbf{G}_{21}\\
\mathbf{G}_6\circ\mathbf{E}_4 &= \mathbf{E}_2\circ\mathbf{G}_{22}\\
\mathbf{G}_7\circ\mathbf{E}_4 &= \mathbf{E}_2\circ\mathbf{G}_{23}
\end{aligned}$$

$$\begin{aligned}
\mathbf{G}_8\circ\mathbf{G}_{24}\circ\mathbf{E}_5 &= \mathbf{G}_8\\
\mathbf{G}_9\circ\mathbf{G}_{24}\circ\mathbf{E}_5 &= \mathbf{B}_1\\
\mathbf{G}_3\circ\mathbf{G}_{25}\circ\mathbf{E}_5 &= \mathbf{G}_9\\
\mathbf{G}_4\circ\mathbf{G}_{25}\circ\mathbf{E}_5 &= \mathbf{E}_1\circ\mathbf{G}_{24}\circ\mathbf{E}_5
\end{aligned}
\qquad
\begin{aligned}
\mathbf{G}_8\circ\mathbf{G}_{24}\circ\mathbf{E}_6 &= \mathbf{G}_8\\
\mathbf{G}_9\circ\mathbf{G}_{24}\circ\mathbf{E}_6 &= \mathbf{E}_1\\
\mathbf{G}_3\circ\mathbf{G}_{25}\circ\mathbf{E}_6 &= \mathbf{B}_1\circ\mathbf{G}_{24}\circ\mathbf{E}_6\\
\mathbf{G}_4\circ\mathbf{G}_{25}\circ\mathbf{E}_6 &= \mathbf{G}_9
\end{aligned}$$

$$
\begin{aligned}
\mathbf{G}_{27} \circ \mathbf{E}_7 &= \mathbf{G}_{32} & \mathbf{G}_{31} \circ \mathbf{G}_{26} \circ \mathbf{E}_8 &= \mathbf{G}_{31} \\
\mathbf{G}_{28} \circ \mathbf{E}_7 &= \mathbf{G}_{25} \circ \mathbf{E}_5 \circ \mathbf{G}_{19} & \mathbf{G}_{32} \circ \mathbf{G}_{26} \circ \mathbf{E}_8 &= \mathbf{E}_2 \\
\mathbf{G}_{29} \circ \mathbf{E}_7 &= \mathbf{G}_{25} \circ \mathbf{E}_5 \circ \mathbf{G}_{18} & \mathbf{G}_{27} \circ \mathbf{E}_8 &= \mathbf{B}_4 \circ \mathbf{G}_{36} \circ \mathbf{B}_9 \circ \mathbf{G}_{26} \circ \mathbf{E}_8 \\
\mathbf{G}_{30} \circ \mathbf{E}_7 &= \mathbf{E}_2 \circ \mathbf{G}_{26} \circ \mathbf{E}_7 & \mathbf{G}_{28} \circ \mathbf{E}_8 &= \mathbf{G}_{25} \circ \mathbf{E}_6 \circ \mathbf{G}_{19} \\
\mathbf{G}_{31} \circ \mathbf{G}_{26} \circ \mathbf{E}_7 &= \mathbf{G}_{31} & \mathbf{G}_{29} \circ \mathbf{E}_8 &= \mathbf{G}_{25} \circ \mathbf{E}_6 \circ \mathbf{G}_{18} \\
\mathbf{G}_{32} \circ \mathbf{G}_{26} \circ \mathbf{E}_7 &= \mathbf{B}_4 \circ \mathbf{G}_{36} \circ \mathbf{B}_9 & \mathbf{G}_{30} \circ \mathbf{E}_8 &= \mathbf{G}_{32}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{G}_{31} \circ \mathbf{G}_{33} \circ \mathbf{E}_9 &= \mathbf{G}_8 & \mathbf{G}_{31} \circ \mathbf{G}_{33} \circ \mathbf{E}_{10} &= \mathbf{G}_8 \\
\mathbf{G}_{32} \circ \mathbf{G}_{33} \circ \mathbf{E}_9 &= \mathbf{G}_{25} \circ \mathbf{E}_6 & \mathbf{G}_{32} \circ \mathbf{G}_{33} \circ \mathbf{E}_{10} &= \mathbf{G}_{25} \circ \mathbf{E}_5 \\
\mathbf{G}_9 \circ \mathbf{G}_{34} \circ \mathbf{E}_9 &= \mathbf{E}_1 & \mathbf{G}_9 \circ \mathbf{G}_{34} \circ \mathbf{E}_{10} &= \mathbf{B}_1 \\
\mathbf{G}_5 \circ \mathbf{G}_{35} \circ \mathbf{E}_9 &= \mathbf{G}_{25} \circ \mathbf{E}_5 \circ \mathbf{G}_{34} \circ \mathbf{E}_9 & \mathbf{G}_5 \circ \mathbf{G}_{35} \circ \mathbf{E}_{10} &= \mathbf{B}_4 \circ \mathbf{G}_{36} \circ \mathbf{B}_9 \circ \mathbf{G}_{33} \circ \mathbf{E}_{10} \\
\mathbf{G}_6 \circ \mathbf{G}_{35} \circ \mathbf{E}_9 &= \mathbf{E}_2 \circ \mathbf{G}_{33} \circ \mathbf{E}_9 & \mathbf{G}_6 \circ \mathbf{G}_{35} \circ \mathbf{E}_{10} &= \mathbf{G}_{25} \circ \mathbf{E}_6 \circ \mathbf{G}_{34} \circ \mathbf{E}_{10} \\
\mathbf{G}_7 \circ \mathbf{G}_{35} \circ \mathbf{E}_9 &= \mathbf{G}_2 \circ \mathbf{E}_1 & \mathbf{G}_7 \circ \mathbf{G}_{35} \circ \mathbf{E}_{10} &= \mathbf{G}_2 \circ \mathbf{B}_1
\end{aligned}
$$

Theorem 4.4.6. There exists a Lawvere $\mathbf{Cat}_o$-theory **Ccc** for which the models are all Cartesian closed categories.

4.4.4 Cat-Enriched Lawvere Cat-Theories

In this section, we analyse the Lawvere $\mathbf{Cat}_o$-Theories in the previous sections, from the point of view of enriched category theory [20]. We extend some of them to **Cat**-enriched one. The models and morphisms remain the same, but we also have 2-cells by natural transformations. We can prove that **Cat** is a locally finitely presentable 2-category. On one hand, **Ct** and **Cb** can be **Cat**-enriched. On the other hand, we expect that **Ccc** can not be **Cat**-enriched.

Theorem 4.4.7 (Categories with terminal objects)**.** Let **Ct2** be the Lawvere **Cat**-theory freely generated from $\mathbf{Cat}_f^{\mathrm{op}}$ by adding the same new arrows and equations as **Ct**. There exists a bijection between the class of all models for **Ct2** and the class of all models for **Ct**.

Proof. Let $\mathbf{ob}\colon \mathbf{Cat} \to \mathbf{Set}$ be the functor that sends a category to the set of the objects. If (C, T) is a model of **Ct2**, then $(C, \mathbf{ob} \circ T)$ is a model of **Ct**.

Conversely, we prove that for model (C, S) of **Ct**, there exists a unique model (C, T) of **Ct2** such that $\mathbf{ob} \circ T = S$. Here, we show that $T\mathbf{T}_1$ and $T\mathbf{T}_2$ are uniquely determined by (C, S). The object part of the functor $T\mathbf{T}_1$ is determined by $S\mathbf{T}_1$. We define an arrow part of the functor $T\mathbf{T}_1$ by the identity arrow on 1. The object part of the functor $T\mathbf{T}_2$ is determined by $S\mathbf{T}_2$. Take an arrow $\alpha\colon x \to x'$ in the category $\mathbf{Cat}(1, C)$. We define an arrow part of the functor $T\mathbf{T}_2$ as follows. Then, $T\mathbf{T}_2$ becomes a functor.

$$\begin{array}{ccc} x & \xrightarrow{!_x} & 1 \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\mathbf{Id}} \\ x' & \xrightarrow[!_{x'}]{} & 1 \end{array}$$

□

Theorem 4.4.8 (Categories with binary products)**.** Let **Cb2** be the Lawvere **Cat**-theory freely generated from $\mathbf{Cat}_f^{\mathrm{op}}$ by adding the same new arrows and equations as **Cb**. There exists a bijection between the class of all models for **Cb2** and the class of all models for **Cb**.

Proof. Take arrows $\alpha\colon x \to x'$ and $\beta\colon y \to y'$ in the category $\mathbf{Cat}(2, C)$. We define an arrow part of a functor for $\mathbf{B}_1$ by $\alpha \times \beta\colon x \times y \to x' \times y'$. □

The algebraic structure for CCC on **Cat** can not be **Cat**-enriched[5]. In fact, we can not define an arrow part of a functor for $\mathbf{E}_1$. (It must send arrows $\alpha\colon x \to x'$ and $\beta\colon y \to y'$ in the category $\mathbf{Cat}(2, C)$ to an arrow from $[x, y]$ to $[x', y']$ in the category $\mathbf{Cat}(1, C)$)

4.5 Example: RMu-Algebras

In this section, we give the Lawvere A-theory **RMu** for **RMu**-algebras. First, putting $V = \mathbf{Set}$ and $A = \mathbf{LocOrd}$, we give the Lawvere **LocOrd**-theory for which the models are all **RMu**-algebras. Next, we extend it to those in the setting $V = \mathbf{Cat}$ and $A = \mathbf{LocOrd}_{lr}$.

4.5.1 Lawvere LocOrd-Theory RMu

In this section, we give a rule-based presentation for **RMu**-algebras. Then, we construct Lawvere **LocOrd**-theory **RMu** from the rule-based presentation, similarly to Section 4.4.

We use judgements for objects, arrows, and inequations between arrows as follows. Let C be an arbitrary locally ordered category. A judgement $x\colon \mathbf{obj}$ represents that x is an object of C. A judgement $f\colon x \to y$ represents that f is an arrow from x to y in C. A judgement $f \Rightarrow g\colon x \to y$ represents that $f, g\colon x \to y$ are arrows of C and that g is greater than f.

$$\frac{x\colon \mathbf{obj} \quad y\colon \mathbf{obj}}{\bot_{x,y}\colon x \to y}\,M_1 \qquad \frac{f\colon x \to y}{\bot_{x,y} \Rightarrow f\colon x \to y}\,M_2$$

$$\frac{f\colon x \to y \quad g\colon x \to y}{f \vee_{x,y} g\colon x \to y}\,M_3 \qquad \frac{f \Rightarrow h\colon x \to y \quad g \Rightarrow h\colon x \to y}{f \vee_{x,y} g \Rightarrow h\colon x \to y}\,M_4$$

$$\frac{f\colon x \to y \quad g\colon x \to y}{f \Rightarrow f \vee_{x,y} g\colon x \to y}\,M_5 \qquad \frac{f\colon x \to y \quad g\colon x \to y}{g \Rightarrow f \vee_{x,y} g\colon x \to y}\,M_6$$

$$\frac{x\colon \mathbf{obj} \quad y\colon \mathbf{obj}}{\top_{x,y}\colon x \to y}\,M_7 \qquad \frac{f\colon x \to y}{f \Rightarrow \top_{x,y}\colon x \to y}\,M_8$$

$$\frac{f\colon x \to y \quad g\colon x \to y}{f \wedge_{x,y} g\colon x \to y}\,M_9 \qquad \frac{h \Rightarrow f\colon x \to y \quad h \Rightarrow g\colon x \to y}{h \Rightarrow f \wedge_{x,y} g\colon x \to y}\,M_{10}$$

$$\frac{f\colon x \to y \quad g\colon x \to y}{f \wedge_{x,y} g \Rightarrow f\colon x \to y}\,M_{11} \qquad \frac{f\colon x \to y \quad g\colon x \to y}{f \wedge_{x,y} g \Rightarrow g\colon x \to y}\,M_{12}$$

$$\frac{f\colon x \to y \quad g\colon y \to y}{\mu(f,g)\colon x \to y}\,M_{13} \qquad \frac{f\colon x \to y \quad g\colon y \to y}{f \Rightarrow \mu(f,g)\colon x \to y}\,M_{14}$$

$$\frac{f\colon x \to y \quad g\colon y \to y}{g \circ \mu(f,g) \Rightarrow \mu(f,g)\colon x \to y}\,M_{15} \qquad \frac{f \Rightarrow h\colon x \to y \quad g \circ h \Rightarrow h\colon x \to y}{\mu(f,g) \Rightarrow h\colon x \to y}\,M_{16}$$

$$\frac{f\colon x \to y \quad g\colon y \to y}{\nu(f,g)\colon x \to y}\,M_{17} \qquad \frac{f\colon x \to y \quad g\colon y \to y}{\nu(f,g) \Rightarrow f\colon x \to y}\,M_{18}$$

$$\frac{f\colon x \to y \quad g\colon y \to y}{\nu(f,g) \Rightarrow g \circ \nu(f,g)\colon x \to y}\,M_{19} \qquad \frac{h \Rightarrow f\colon x \to y \quad h \Rightarrow g \circ h\colon x \to y}{h \Rightarrow \nu(f,g)\colon x \to y}\,M_{20}$$

$$\frac{f\colon x \to y \quad z\colon \mathbf{obj}}{\bot_{y,z} \circ f \Rightarrow \bot_{x,z}\colon x \to z}\,M_{21} \qquad \frac{f\colon y \to z \quad g\colon y \to z \quad h\colon x \to y}{(f \vee_{y,z} g) \circ h \Rightarrow (f \circ h) \vee_{x,z} (g \circ h)\colon x \to z}\,M_{22}$$

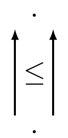
$$\frac{f\colon x \to y \quad z\colon \mathbf{obj}}{\top_{x,z} \Rightarrow \top_{y,z} \circ f\colon x \to z}\,M_{23} \qquad \frac{f\colon y \to z \quad g\colon y \to z \quad h\colon x \to y}{(f \circ h) \wedge_{x,z} (g \circ h) \Rightarrow (f \wedge_{y,z} g) \circ h\colon x \to z}\,M_{24}$$

$$\frac{h\colon x \to y \quad f\colon y \to z \quad g\colon z \to z}{\mu(f,g) \circ h \Rightarrow \mu(f \circ h, g)\colon x \to z}\,M_{25} \qquad \frac{h\colon x \to y \quad f\colon y \to z \quad g\colon z \to z}{\nu(f \circ h, g) \Rightarrow \nu(f,g) \circ h\colon x \to z}\,M_{26}$$

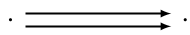
Theorem 4.5.1. A locally ordered small category C satisfies M_1 through M_{26} if and only if C is an **RMu**-algebra.

We can check that **LocOrd** is a locally finitely presentable category. For later use, we name some finitely presentable objects in **LocOrd**.

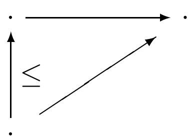
- locally ordered categories freely generated from finitely presentable objects in $\mathbf{Cat}_o$
- $\mathbf{A_3}$: the locally ordered category freely generated from the following locally ordered graph.



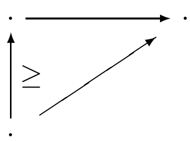
- $\mathbf{A_4}$: the locally ordered category freely generated from the following locally ordered graph.



- $\mathbf{A_5}$: the locally ordered category freely generated from the following locally ordered graph.

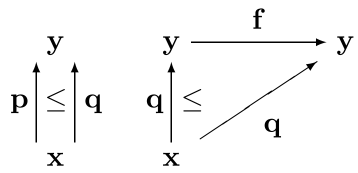


- $\mathbf{A_6}$: the locally ordered category freely generated from the following locally ordered graph.

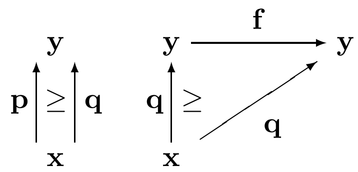


- $\mathbf{A_7}$: the locally ordered category freely generated from $\mathbf{p}\colon \mathbf{x} \to \mathbf{y}$ and $\mathbf{f}\colon \mathbf{y} \to \mathbf{y}$.
- $\mathbf{A_8}$: the locally ordered category freely generated from $\mathbf{s}\colon \mathbf{z} \to \mathbf{x}$, $\mathbf{p}\colon \mathbf{x} \to \mathbf{y}$, and $\mathbf{f}\colon \mathbf{y} \to \mathbf{y}$.

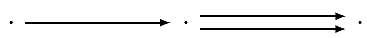
- $\mathbf{A_9}$: the locally ordered category freely generated from $\mathbf{p}, \mathbf{q} \colon \mathbf{x} \to \mathbf{y}$ and $\mathbf{f} \colon \mathbf{y} \to \mathbf{y}$ subject to inequalities $\mathbf{p} \leq \mathbf{q}$ and $\mathbf{f} \circ \mathbf{q} \leq \mathbf{q}$.



- $\mathbf{A_{10}}$: the locally ordered category freely generated from $\mathbf{p}, \mathbf{q} \colon \mathbf{x} \to \mathbf{y}$ and $\mathbf{f} \colon \mathbf{y} \to \mathbf{y}$ subject to inequalities $\mathbf{q} \leq \mathbf{p}$ and $\mathbf{q} \leq \mathbf{f} \circ \mathbf{q}$.



- $\mathbf{A_{11}}$: the locally ordered category freely generated from the following locally ordered graph. (This diagram does not commute)



- $\mathbf{A_{12}}$: the locally ordered category freely generated from $\mathbf{p}, \mathbf{q}, \mathbf{r} \colon \mathbf{x} \to \mathbf{y}$ subject to inequalities $\mathbf{p} \leq \mathbf{r}$ and $\mathbf{q} \leq \mathbf{r}$.

- $\mathbf{A_{13}}$: the locally ordered category freely generated from $\mathbf{p}, \mathbf{q}, \mathbf{r} \colon \mathbf{x} \to \mathbf{y}$ subject to inequalities $\mathbf{r} \leq \mathbf{p}$ and $\mathbf{r} \leq \mathbf{q}$.

Let **RMu0** be the freely generated Lawvere **LocOrd**-theory from $\mathbf{LocOrd}_f^{\mathrm{op}}$ by adding the new arrows called as follows.

		$\mathrm{M}_{13} \colon \mathbf{A_7} \to \mathbf{2}$	
$\mathrm{M}_1 \colon 2 \to \mathbf{2}$	$\mathrm{M}_7 \colon 2 \to \mathbf{2}$	$\mathrm{M}_{14} \colon \mathbf{A_7} \to \mathbf{A_3}$	$\mathrm{M}_{21} \colon 1 + \mathbf{2} \to \mathbf{A_5}$
$\mathrm{M}_2 \colon \mathbf{2} \to \mathbf{A_3}$	$\mathrm{M}_8 \colon \mathbf{2} \to \mathbf{A_3}$	$\mathrm{M}_{15} \colon \mathbf{A_7} \to \mathbf{A_5}$	$\mathrm{M}_{22} \colon \mathbf{A_{11}} \to \mathbf{A_5}$
$\mathrm{M}_3 \colon \mathbf{A_4} \to \mathbf{2}$	$\mathrm{M}_9 \colon \mathbf{A_4} \to \mathbf{2}$	$\mathrm{M}_{16} \colon \mathbf{A_9} \to \mathbf{A_3}$	$\mathrm{M}_{23} \colon 1 + \mathbf{2} \to \mathbf{A_6}$
$\mathrm{M}_4 \colon \mathbf{A_{12}} \to \mathbf{A_3}$	$\mathrm{M}_{10} \colon \mathbf{A_{13}} \to \mathbf{A_3}$	$\mathrm{M}_{17} \colon \mathbf{A_7} \to \mathbf{2}$	$\mathrm{M}_{24} \colon \mathbf{A_{11}} \to \mathbf{A_6}$
$\mathrm{M}_5 \colon \mathbf{A_4} \to \mathbf{A_3}$	$\mathrm{M}_{11} \colon \mathbf{A_4} \to \mathbf{A_3}$	$\mathrm{M}_{18} \colon \mathbf{A_7} \to \mathbf{A_3}$	$\mathrm{M}_{25} \colon \mathbf{A_8} \to \mathbf{A_5}$
$\mathrm{M}_6 \colon \mathbf{A_4} \to \mathbf{A_3}$	$\mathrm{M}_{12} \colon \mathbf{A_4} \to \mathbf{A_3}$	$\mathrm{M}_{19} \colon \mathbf{A_7} \to \mathbf{A_6}$	$\mathrm{M}_{26} \colon \mathbf{A_8} \to \mathbf{A_6}$
		$\mathrm{M}_{20} \colon \mathbf{A_{10}} \to \mathbf{A_3}$	

Next, we give certain equations among arrows of the Lawvere **LocOrd**-theory **RMu0**. Let (C, S) be a model of **RMu0**.

There exists $\mathbf{G}_{38}\colon \mathbf{A_4} \to 2$ such that for arrow $f, g\colon x \to y$ in C, the function $S\mathbf{G}_{38}$ returns x and y.

$$\frac{f\colon x \to y \quad g\colon x \to y}{x, y\colon \mathbf{obj}}\ \mathbf{G}_{38}$$

There exist $\mathbf{G}_{39}, \mathbf{G}_{40}, \mathbf{G}_{41}\colon \mathbf{A_5} \to \mathbf{2}$ such that for arrows $f\colon x \to y$, $g\colon y \to z$, and $h\colon x \to z$ in C such that $g \circ f \leq h$, the functions $S\mathbf{G}_{39}, S\mathbf{G}_{40}, S\mathbf{G}_{41}$ return f, g, and h, respectively.

$$\frac{f\colon x \to y \quad g\colon y \to z \quad g \circ f \Rightarrow h\colon x \to z}{f\colon x \to y}\ \mathbf{G}_{39}$$
$$\frac{f\colon x \to y \quad g\colon y \to z \quad g \circ f \Rightarrow h\colon x \to z}{g\colon y \to z}\ \mathbf{G}_{40}$$
$$\frac{f\colon x \to y \quad g\colon y \to z \quad g \circ f \Rightarrow h\colon x \to z}{h\colon x \to z}\ \mathbf{G}_{41}$$

There exist $\mathbf{G}_{42}, \mathbf{G}_{43}, \mathbf{G}_{44}\colon \mathbf{A_6} \to \mathbf{2}$ such that for arrows $f\colon x \to y$, $g\colon y \to z$, and $h\colon x \to z$ in C such that $h \leq g \circ f$, the functions $S\mathbf{G}_{42}, S\mathbf{G}_{43}, S\mathbf{G}_{44}$ return f, g, and h, respectively.

$$\frac{f\colon x \to y \quad g\colon y \to z \quad h \Rightarrow g \circ f\colon x \to z}{f\colon x \to y}\ \mathbf{G}_{42}$$
$$\frac{f\colon x \to y \quad g\colon y \to z \quad h \Rightarrow g \circ f\colon x \to z}{g\colon y \to z}\ \mathbf{G}_{43}$$
$$\frac{f\colon x \to y \quad g\colon y \to z \quad h \Rightarrow g \circ f\colon x \to z}{h\colon x \to z}\ \mathbf{G}_{44}$$

There exists $\mathbf{G}_{45}\colon \mathbf{A_7} \to 2$ such that for arrows $f\colon x \to y$ and $g\colon y \to y$ in C, the function $S\mathbf{G}_{45}$ returns x and y.

$$\frac{f\colon x \to y \quad g\colon y \to y}{x, y\colon \mathbf{obj}}\ \mathbf{G}_{45}$$

There exists $\mathbf{G}_{46}\colon \mathbf{A_7} \to \mathbf{2}$ such that for arrows $f\colon x \to y$ and $g\colon y \to y$ in C, the function $S\mathbf{G}_{46}$ returns f.

$$\frac{f\colon x \to y \quad g\colon y \to y}{f\colon x \to y}\ \mathbf{G}_{46}$$

There exists $\mathbf{G}_{47}\colon \mathbf{A_7} \to \mathbf{2}$ such that for arrows $f\colon x \to y$ and $g\colon y \to y$ in C, the function $S\mathbf{G}_{47}$ returns g.

$$\frac{f\colon x \to y \quad g\colon y \to y}{g\colon y \to y}\ \mathbf{G}_{47}$$

There exists $\mathbf{G}_{48}\colon \mathbf{2} \to 2$ such that for arrow $f\colon x \to y$ in C, the function $S\mathbf{G}_{48}$ returns x and y.

$$\frac{f\colon x \to y}{x, y\colon \mathbf{obj}}\ \mathbf{G}_{48}$$

There exist $\mathbf{G}_{49}, \mathbf{G}_{50}\colon \mathbf{A_3} \to \mathbf{2}$ such that for arrows $f\colon x \to y$ and $g\colon x \to y$ in C such that $f \leq g$, the functions $S\mathbf{G}_{49}$ and $S\mathbf{G}_{50}$ return f and g, respectively.

$$\frac{f \Rightarrow g\colon x \to y}{f\colon x \to y}\ \mathbf{G}_{49} \quad \frac{f \Rightarrow g\colon x \to y}{g\colon x \to y}\ \mathbf{G}_{50}$$

There exists $\mathbf{G}_{51}\colon \mathbf{A_9} \to \mathbf{2}$ such that for arrows $f\colon x \to y$, $g\colon y \to y$, and $h\colon x \to y$ in C such that $f \leq h$ and $g \circ h \leq h$, the function $S\mathbf{G}_{51}$ returns h.

$$\frac{f \Rightarrow h\colon x \to y \quad g \circ h \Rightarrow h\colon x \to y}{h\colon x \to y}\ \mathbf{G}_{51}$$

There exists $\mathbf{G}_{52}\colon \mathbf{A_9} \to \mathbf{A_7}$ such that for arrows $f\colon x \to y$, $g\colon y \to y$, and $h\colon x \to y$ in C such that $f \leq h$ and $g \circ h \leq h$, the function $S\mathbf{G}_{52}$ returns f and g.

$$\frac{f \Rightarrow h\colon x \to y \quad g \circ h \Rightarrow h\colon x \to y}{f\colon x \to y \quad g\colon y \to y}\ \mathbf{G}_{52}$$

There exists $\mathbf{G}_{53}\colon \mathbf{A_8} \to \mathbf{3}$ such that for arrows $f\colon x \to y$, $g\colon y \to y$, and $h\colon z \to x$ in C, the function $S\mathbf{G}_{53}$ returns h.

$$\frac{f\colon x \to y \quad g\colon y \to y \quad h\colon z \to x}{h\colon z \to x}\ \mathbf{G}_{53}$$

There exists $\mathbf{G}_{54}\colon \mathbf{A_8} \to \mathbf{A_7}$ such that for arrows $f\colon x \to y$, $g\colon y \to y$, and $h\colon z \to x$ in C, the function $S\mathbf{G}_{54}$ returns $f\colon x \to y$ and $g\colon y \to y$.

$$\frac{f\colon x \to y \quad g\colon y \to y \quad h\colon z \to x}{f\colon x \to y \quad g\colon y \to y}\ \mathbf{G}_{54}$$

There exists $\mathbf{G}_{55}\colon \mathbf{A_8} \to \mathbf{A_7}$ such that for arrows $f\colon x \to y$, $g\colon y \to y$, and $h\colon z \to x$ in C, the function $S\mathbf{G}_{55}$ returns $f \circ h\colon z \to y$ and $g\colon y \to y$.

$$\frac{f\colon x \to y \quad g\colon y \to y \quad h\colon z \to x}{f \circ h\colon z \to y \quad g\colon y \to y}\ \mathbf{G}_{55}$$

There exists $\mathbf{G}_{56}\colon \mathbf{A_{10}} \to \mathbf{2}$ such that for arrows $f\colon x \to y$, $g\colon y \to y$, and $h\colon x \to y$ in C such that $h \leq f$ and $h \leq g \circ h$, the function $S\mathbf{G}_{56}$ returns h.

$$\frac{h \Rightarrow f\colon x \to y \quad h \Rightarrow g \circ h\colon x \to y}{h\colon x \to y}\ \mathbf{G}_{56}$$

There exists $\mathbf{G}_{57}\colon \mathbf{A_{10}} \to \mathbf{A_7}$ such that for arrows $f\colon x \to y$, $g\colon y \to y$, and $h\colon x \to y$ in C such that $h \circ f$ and $h \leq g \circ h$, the function $S\mathbf{G}_{57}$ returns f and g.

$$\frac{h \Rightarrow f\colon x \to y \quad h \Rightarrow g \circ h\colon x \to y}{f\colon x \to y \quad g\colon y \to y}\ \mathbf{G}_{57}$$

There exist $\mathbf{G}_{58}, \mathbf{G}_{59}\colon \mathbf{A_4} \to \mathbf{2}$ such that for arrows $f\colon x \to y$ and $g\colon x \to y$ in C, the functions $S\mathbf{G}_{58}$ and $S\mathbf{G}_{59}$ return f and g, respectively.

$$\frac{f\colon x \to y \quad g\colon x \to y}{f\colon x \to y}\ \mathbf{G}_{58} \quad \frac{f\colon x \to y \quad g\colon x \to y}{g\colon x \to y}\ \mathbf{G}_{59}$$

There exists $\mathbf{G}_{60}\colon \mathbf{A_{12}} \to \mathbf{2}$ such that for arrows $f\colon x \to y$, $g\colon x \to y$, and $h\colon x \to y$ in C such that $f \leq h$ and $g \leq h$, the function $S\mathbf{G}_{60}$ returns h.

$$\frac{f \Rightarrow h\colon x \to y \quad g \Rightarrow h\colon x \to y}{h\colon x \to y}\ \mathbf{G}_{60}$$

There exists $\mathbf{G}_{61}\colon \mathbf{A_{12}} \to \mathbf{A_4}$ such that for arrows $f\colon x \to y$, $g\colon x \to y$, and $h\colon x \to y$ in C such that $f \leq h$ and $g \leq h$, the function $S\mathbf{G}_{61}$ returns f and g.

$$\frac{f \Rightarrow h\colon x \to y \quad g \Rightarrow h\colon x \to y}{f\colon x \to y \quad g\colon x \to y}\ \mathbf{G}_{61}$$

There exists $\mathbf{G}_{62}\colon \mathbf{A_{13}} \to \mathbf{2}$ such that for arrows $f\colon x \to y$, $g\colon x \to y$, and $h\colon x \to y$ in C such that $h \leq f$ and $h \leq g$, the function $S\mathbf{G}_{62}$ returns h.

$$\frac{h \Rightarrow f\colon x \to y \quad h \Rightarrow g\colon x \to y}{h\colon x \to y}\ \mathbf{G}_{62}$$

There exists $\mathbf{G}_{63}\colon \mathbf{A_{13}} \to \mathbf{A_4}$ such that for arrows $f\colon x \to y$, $g\colon x \to y$, and $h\colon x \to y$ in C such that $h \leq f$ and $h \leq g$, the function $S\mathbf{G}_{63}$ returns f and g.

$$\frac{h \Rightarrow f\colon x \to y \quad h \Rightarrow g\colon x \to y}{f\colon x \to y \quad g\colon x \to y}\ \mathbf{G}_{63}$$

There exists $\mathbf{G}_{64}\colon 1 + \mathbf{2} \to 2$ such that for arrow $f\colon x \to y$ and object z in C, the function $S\mathbf{G}_{64}$ returns y and z.

$$\frac{f\colon x \to y \quad z\colon \mathbf{obj}}{y, z\colon \mathbf{obj}}\ \mathbf{G}_{64}$$

There exists $\mathbf{G}_{65}\colon 1+\mathbf{2}\to 2$ such that for arrow $f\colon x\to y$ and object z in C, the function $S\mathbf{G}_{65}$ returns x and z.

$$\frac{f\colon x\to y \quad z\colon \mathbf{obj}}{x,z\colon \mathbf{obj}}\ \mathbf{G}_{65}$$

There exists $\mathbf{G}_{66}\colon \mathbf{A_{11}}\to\mathbf{2}$ such that for arrows $f\colon y\to z$, $g\colon y\to z$, and $h\colon x\to y$ in C, the function $S\mathbf{G}_{66}$ returns h.

$$\frac{f,g\colon y\to z \quad h\colon x\to y}{h\colon x\to y}\ \mathbf{G}_{66}$$

There exists $\mathbf{G}_{67}\colon \mathbf{A_{11}}\to\mathbf{A_4}$ such that for arrows $f\colon y\to z$, $g\colon y\to z$, and $h\colon x\to y$ in C, the function $S\mathbf{G}_{67}$ returns f and g.

$$\frac{f,g\colon y\to z \quad h\colon x\to y}{f\colon y\to z \quad g\colon y\to z}\ \mathbf{G}_{67}$$

There exists $\mathbf{G}_{68}\colon \mathbf{A_{11}}\to\mathbf{A_4}$ such that for arrows $f\colon y\to z$, $g\colon y\to z$, and $h\colon x\to y$ in C, the function $S\mathbf{G}_{68}$ returns $f\circ h$ and $g\circ h$.

$$\frac{f,g\colon y\to z \quad h\colon x\to y}{f\circ h\colon x\to z \quad g\circ h\colon x\to z}\ \mathbf{G}_{68}$$

Let **RMu** be the freely generated Lawvere **LocOrd**-theory from **RMu0** subject to the following equations.

$$\begin{array}{rcl@{\qquad}rcl}
\mathbf{G}_{48}\circ\mathbf{M}_1 &=& \mathbf{Id} & \mathbf{G}_{48}\circ\mathbf{M}_7 &=& \mathbf{Id}\\
\mathbf{G}_{49}\circ\mathbf{M}_2 &=& \mathbf{M}_1\circ\mathbf{G}_{48} & \mathbf{G}_{49}\circ\mathbf{M}_8 &=& \mathbf{Id}\\
\mathbf{G}_{50}\circ\mathbf{M}_2 &=& \mathbf{Id} & \mathbf{G}_{50}\circ\mathbf{M}_8 &=& \mathbf{M}_7\circ\mathbf{G}_{48}\\
\mathbf{G}_{48}\circ\mathbf{M}_3 &=& \mathbf{G}_{38} & \mathbf{G}_{48}\circ\mathbf{M}_9 &=& \mathbf{G}_{38}\\
\mathbf{G}_{49}\circ\mathbf{M}_4 &=& \mathbf{M}_3\circ\mathbf{G}_{61} & \mathbf{G}_{49}\circ\mathbf{M}_{10} &=& \mathbf{G}_{62}\\
\mathbf{G}_{50}\circ\mathbf{M}_4 &=& \mathbf{G}_{60} & \mathbf{G}_{50}\circ\mathbf{M}_{10} &=& \mathbf{M}_9\circ\mathbf{G}_{63}\\
\mathbf{G}_{49}\circ\mathbf{M}_5 &=& \mathbf{G}_{58} & \mathbf{G}_{49}\circ\mathbf{M}_{11} &=& \mathbf{M}_9\\
\mathbf{G}_{50}\circ\mathbf{M}_5 &=& \mathbf{M}_3 & \mathbf{G}_{50}\circ\mathbf{M}_{11} &=& \mathbf{G}_{58}\\
\mathbf{G}_{49}\circ\mathbf{M}_6 &=& \mathbf{G}_{59} & \mathbf{G}_{49}\circ\mathbf{M}_{12} &=& \mathbf{M}_9\\
\mathbf{G}_{50}\circ\mathbf{M}_6 &=& \mathbf{M}_3 & \mathbf{G}_{50}\circ\mathbf{M}_{12} &=& \mathbf{G}_{59}
\end{array}$$

$$
\begin{array}{rclrcl}
\mathbf{G}_{48} \circ \mathbf{M}_{13} & = & \mathbf{G}_{45} & \mathbf{G}_{48} \circ \mathbf{M}_{17} & = & \mathbf{G}_{45} \\
\mathbf{G}_{49} \circ \mathbf{M}_{14} & = & \mathbf{G}_{46} & \mathbf{G}_{49} \circ \mathbf{M}_{18} & = & \mathbf{M}_{17} \\
\mathbf{G}_{50} \circ \mathbf{M}_{14} & = & \mathbf{M}_{13} & \mathbf{G}_{50} \circ \mathbf{M}_{18} & = & \mathbf{G}_{46} \\
\mathbf{G}_{39} \circ \mathbf{M}_{15} & = & \mathbf{M}_{13} & \mathbf{G}_{42} \circ \mathbf{M}_{19} & = & \mathbf{M}_{17} \\
\mathbf{G}_{40} \circ \mathbf{M}_{15} & = & \mathbf{G}_{47} & \mathbf{G}_{43} \circ \mathbf{M}_{19} & = & \mathbf{G}_{47} \\
\mathbf{G}_{41} \circ \mathbf{M}_{15} & = & \mathbf{M}_{13} & \mathbf{G}_{44} \circ \mathbf{M}_{19} & = & \mathbf{M}_{17} \\
\mathbf{G}_{49} \circ \mathbf{M}_{16} & = & \mathbf{M}_{13} \circ \mathbf{G}_{52} & \mathbf{G}_{49} \circ \mathbf{M}_{20} & = & \mathbf{G}_{56} \\
\mathbf{G}_{50} \circ \mathbf{M}_{16} & = & \mathbf{G}_{51} & \mathbf{G}_{50} \circ \mathbf{M}_{20} & = & \mathbf{M}_{17} \circ \mathbf{G}_{57}
\end{array}
$$

$$
\begin{array}{rclrcl}
\mathbf{G}_{39} \circ \mathbf{M}_{21} & = & \mathbf{G}_{32} & & & \\
\mathbf{G}_{40} \circ \mathbf{M}_{21} & = & \mathbf{M}_{1} \circ \mathbf{G}_{64} & & & \\
\mathbf{G}_{41} \circ \mathbf{M}_{21} & = & \mathbf{M}_{1} \circ \mathbf{G}_{65} & & & \\
\mathbf{G}_{39} \circ \mathbf{M}_{22} & = & \mathbf{G}_{66} & \mathbf{G}_{39} \circ \mathbf{M}_{25} & = & \mathbf{G}_{53} \\
\mathbf{G}_{40} \circ \mathbf{M}_{22} & = & \mathbf{M}_{3} \circ \mathbf{G}_{67} & \mathbf{G}_{40} \circ \mathbf{M}_{25} & = & \mathbf{M}_{13} \circ \mathbf{G}_{54} \\
\mathbf{G}_{41} \circ \mathbf{M}_{22} & = & \mathbf{M}_{3} \circ \mathbf{G}_{68} & \mathbf{G}_{41} \circ \mathbf{M}_{25} & = & \mathbf{M}_{13} \circ \mathbf{G}_{55} \\
\mathbf{G}_{42} \circ \mathbf{M}_{23} & = & \mathbf{G}_{32} & \mathbf{G}_{42} \circ \mathbf{M}_{26} & = & \mathbf{G}_{53} \\
\mathbf{G}_{43} \circ \mathbf{M}_{23} & = & \mathbf{M}_{7} \circ \mathbf{G}_{64} & \mathbf{G}_{43} \circ \mathbf{M}_{26} & = & \mathbf{M}_{17} \circ \mathbf{G}_{54} \\
\mathbf{G}_{44} \circ \mathbf{M}_{23} & = & \mathbf{M}_{7} \circ \mathbf{G}_{65} & \mathbf{G}_{44} \circ \mathbf{M}_{26} & = & \mathbf{M}_{17} \circ \mathbf{G}_{55} \\
\mathbf{G}_{42} \circ \mathbf{M}_{24} & = & \mathbf{G}_{66} & & & \\
\mathbf{G}_{43} \circ \mathbf{M}_{24} & = & \mathbf{M}_{9} \circ \mathbf{G}_{67} & & & \\
\mathbf{G}_{44} \circ \mathbf{M}_{24} & = & \mathbf{M}_{9} \circ \mathbf{G}_{68} & & &
\end{array}
$$

Theorem 4.5.2. There exists a Lawvere **LocOrd**-theory **RMu** for which the models are all **RMu**-algebras.

This theorem implies Theorem 2.2.3 that the forgetful functor from **RMu-Alg** to **LocOrd** has a left adjoint as a corollary of Theorem 4.2.2.

4.5.2 Cat-Enriched Lawvere $\mathbf{LocOrd}_{lr}$-Theory ERMu

Similarly to Section 4.4.4, we extend Lawvere **LocOrd**-theory **RMu** to the **Cat**-enriched one. Similarly to the paper [23], we can prove that $\mathbf{LocOrd}_{lr}$ is a locally finitely presentable 2-category. Let **ERMu** be the Lawvere $\mathbf{LocOrd}_{lr}$-theory freely generated from $(\mathbf{LocOrd}_{lr})_f^{\mathrm{op}}$ by adding the same arrows and diagrams as those for **RMu**.

Theorem 4.5.3. There exists a bijection between the class of all models for **ERMu** and the class of all models for **RMu**.

Proof. Let $\mathbf{ob}\colon \mathbf{Cat} \to \mathbf{Set}$ be the functor that sends a category to the set of the objects. If $(C, S\colon \mathbf{ERMu} \to \mathbf{Cat})$ is a model of **ERMu**, then $(C, \mathbf{ob} \circ S\colon \mathbf{RMu} \to \mathbf{Set})$ is a model of **RMu**.

Conversely, given a model $(C, S\colon \mathbf{RMu} \to \mathbf{Set})$ of **RMu**, there exists an unique model $(C, T\colon \mathbf{ERMu} \to \mathbf{Cat})$ of **ERMu** such that $\mathbf{ob} \circ T = S$. Here we show that $T\mathbf{M}_{13}\colon \mathbf{LocOrd}_{lr}(\mathbf{A_7}, C) \to \mathbf{LocOrd}_{lr}(\mathbf{2}, C)$ is uniquely determined.

Let $\gamma\colon G \to G'$ be an arrow in $\mathbf{LocOrd}_{lr}(\mathbf{A_7}, C)$ and $\mathbf{2}$ be the category $\{\mathbf{a} \xrightarrow{\mathbf{s}} \mathbf{b}\}$. Writing out lax naturality of γ, we have

$$\begin{array}{ccc} G\mathbf{x} & \xrightarrow{G\mathbf{p}} & G\mathbf{y} \\ \downarrow{\scriptstyle \gamma_{\mathbf{x}}} & \leq & \downarrow{\scriptstyle \gamma_{\mathbf{y}}} \\ G'\mathbf{x} & \xrightarrow[G'\mathbf{p}]{} & G'\mathbf{y} \end{array} \qquad \begin{array}{ccc} G\mathbf{y} & \xrightarrow{G\mathbf{f}} & G\mathbf{y} \\ \downarrow{\scriptstyle \gamma_{\mathbf{y}}} & \leq & \downarrow{\scriptstyle \gamma_{\mathbf{y}}} \\ G'\mathbf{y} & \xrightarrow[G'\mathbf{f}]{} & G'\mathbf{y} \end{array}$$

We need to define $(T\mathbf{M}_{13})\gamma$ that is lax natural, i.e.,

$$\begin{array}{ccc} ((T\mathbf{M}_{13})G)\mathbf{a} & \xrightarrow{((T\mathbf{M}_{13})G)\mathbf{s}} & ((T\mathbf{M}_{13})G)\mathbf{b} \\ {\scriptstyle ((T\mathbf{M}_{13})\gamma)_{\mathbf{a}}}\downarrow & \leq & \downarrow{\scriptstyle ((T\mathbf{M}_{13})\gamma)_{\mathbf{b}}} \\ ((T\mathbf{M}_{13})G')\mathbf{a} & \xrightarrow[((T\mathbf{M}_{13})G')\mathbf{s}]{} & ((T\mathbf{M}_{13})G')\mathbf{b} \end{array}$$

The object part of $T\mathbf{M}_{13}$ must be that of $S\mathbf{M}_{13}$, so $((T\mathbf{M}_{13})G)\mathbf{s} = ((S\mathbf{M}_{13})G)\mathbf{s}$ and $((T\mathbf{M}_{13})G')\mathbf{s} = ((S\mathbf{M}_{13})G')\mathbf{s}$. In the **Cat**-enrichment, the diagrams represent not only equations for the object part, but also ones for the arrow part. So we must define not only that $((T\mathbf{M}_{13})G)\mathbf{a} = G\mathbf{x}$ and $((T\mathbf{M}_{13})G')\mathbf{a} = G'\mathbf{x}$ but also that $((T\mathbf{M}_{13})\gamma)_{\mathbf{a}} = \gamma_{\mathbf{x}}$; similarly, $((T\mathbf{M}_{13})\gamma)_{\mathbf{b}} = \gamma_{\mathbf{y}}$.

It remains to verify that $(T\mathbf{M}_{13})\gamma$ thus defined is lax natural, that is:

$$\begin{array}{ccc} G\mathbf{x} & \xrightarrow{((S\mathbf{M}_{13})G)\mathbf{s}} & G\mathbf{y} \\ \downarrow{\scriptstyle \gamma_{\mathbf{x}}} & \leq & \downarrow{\scriptstyle \gamma_{\mathbf{y}}} \\ G'\mathbf{x} & \xrightarrow[((S\mathbf{M}_{13})G')\mathbf{s}]{} & G'\mathbf{y} \end{array}$$

This is equivalent to $((S\mathbf{M}_{13})G')\mathbf{s} \leq \gamma_{\mathbf{y}} \circ ((S\mathbf{M}_{13})G)\mathbf{s} \circ \alpha_{\mathbf{x}}$ where $\alpha_{\mathbf{x}}$ is the left adjoint to $\gamma_{\mathbf{x}}$. So we are done by $\mathbf{M}_{16}$ if $G'\mathbf{p} \leq$ (RHS) and $G'\mathbf{f} \circ$ (RHS) $\leq$ (RHS). Indeed, we have that

$$\begin{aligned} G'\mathbf{p} &\leq G'\mathbf{p} \circ \gamma_{\mathbf{x}} \circ \alpha_{\mathbf{x}} && \text{(by adjointness)} \\ &\leq \gamma_{\mathbf{y}} \circ G\mathbf{p} \circ \alpha_{\mathbf{x}} && \text{(by lax naturality with respect to } \mathbf{p}) \\ &\leq \gamma_{\mathbf{y}} \circ ((S\mathbf{M}_{13})G)\mathbf{s} \circ \alpha_{\mathbf{x}} && \text{(by } \mathbf{M}_{14}) \end{aligned}$$

and that

$$\begin{aligned} & G'\mathbf{f} \circ \gamma_{\mathbf{y}} \circ ((S\mathbf{M}_{13})G)\mathbf{s} \circ \alpha_{\mathbf{x}} \\ &\quad \leq \gamma_{\mathbf{y}} \circ G\mathbf{f} \circ ((S\mathbf{M}_{13})G)\mathbf{s} \circ \alpha_{\mathbf{x}} && \text{(by lax naturality with respect to } \mathbf{f}) \\ &\quad \leq \gamma_{\mathbf{y}} \circ ((S\mathbf{M}_{13})G)\mathbf{s} \circ \alpha_{\mathbf{x}} && \text{(by } \mathbf{M}_{15}) \end{aligned}$$

□

This theorem implies the theorem that the forgetful 2-functor from $\mathbf{RMu\text{-}Alg}_{lr}$ to $\mathbf{LocOrd}_{lr}$ has a left 2-adjoint as a corollary of Theorem 4.2.2.

4.6 Example: GLTS-Algebras

In this section, we give an algebraic structure for which the models are all **GLTS**-algebras. The structure of **GLTS**-algebras is based on cloven fibrations with structures (see Definition 3.3.1). First, putting $V = \mathbf{Set}$ and $A = \mathbf{Cat}_o^{\rightarrow}$, we give Lawvere $\mathbf{Cat}_o^{\rightarrow}$-theories for which the models are all cloven fibrations with structures. Next, we extend some of them to those in the setting $V = \mathbf{Cat}$ and $A = \mathbf{Cat}^{\rightarrow}$.

4.6.1 Rule-Based Presentation of Cloven Fibrations

In this section, we give a rule-based presentation for cloven fibrations.

We define properties F_1 through F_6 for functors and show rule-based presentations for them. A judgement $\Gamma\colon$ **base** represents that Γ is an object of B. A judgement $u\colon \Gamma \to \Delta$ represents that u is an arrow from Γ to Δ in B. A judgement $\Gamma \vdash \psi$ represents that ψ is an object of E such that $p\psi = \Gamma$. A judgement $u\colon \Gamma \to \Delta \vdash f\colon \psi \to \varphi$ represents that $f\colon \psi \to \varphi$ is an arrow of E such that

$p\psi = \Gamma$, $p\varphi = \Delta$, and $pf = u$. A judgement $u\colon \Gamma \to \Delta \vdash f = g\colon \psi \to \varphi$ represents that $f, g\colon \psi \to \varphi$ are arrows of E such that $p\psi = \Gamma$, $p\varphi = \Delta$, $pf = u$, $pg = u$, and $f = g$. We write $\Gamma \vdash f\colon \psi \to \varphi$ for $\mathbf{Id}\colon \Gamma \to \Gamma \vdash f\colon \psi \to \varphi$.

A functor p satisfies F_1 if for arrow $u\colon \Gamma \to \Delta$ in B and object ψ in E such that $p\psi = \Delta$, there exists an object $u^*\psi$ in E such that $p(u^*\psi) = \Gamma$.

$$\frac{u\colon \Gamma \to \Delta \quad \Delta \vdash \psi}{\Gamma \vdash u^*\psi}\, F_1$$

A functor p satisfies F_2 if for arrow $u\colon \Gamma \to \Delta$ in B and object ψ in E such that $p\psi = \Delta$, there exists an arrow $\overline{u}(\psi)\colon u^*\psi \to \psi$ in E such that $p(\overline{u}(\psi)) = u$.

$$\frac{u\colon \Gamma \to \Delta \quad \Delta \vdash \psi}{u\colon \Gamma \to \Delta \vdash \overline{u}(\psi)\colon u^*\psi \to \psi}\, F_2$$

A functor p satisfies F_3 if for arrow $v\colon \Lambda \to \Gamma$, $u\colon \Gamma \to \Delta$ in B and arrow $f\colon \varphi \to \psi$ in E such that $pf = u \circ v$, there exists an arrow $u_v[f]\colon \varphi \to u^*\psi$ in E such that $p(u_v[f]) = v$.

$$\frac{v\colon \Lambda \to \Gamma \quad u\colon \Gamma \to \Delta \quad u \circ v\colon \Lambda \to \Delta \vdash f\colon \varphi \to \psi}{v\colon \Lambda \to \Gamma \vdash u_v[f]\colon \varphi \to u^*\psi}\, F_3$$

A functor p satisfies F_4 if for arrow $u\colon \Gamma \to \Delta$ in B and object ψ in E such that $p\psi = \Delta$, the arrow $u_{\mathbf{Id}}[\overline{u}(\psi)]\colon u^*\psi \to u^*\psi$ is equal to the identity on $u^*\psi$.

$$\frac{u\colon \Gamma \to \Delta \quad \Delta \vdash \psi}{\mathbf{Id}\colon \Gamma \to \Gamma \vdash \mathbf{Id} = u_{\mathbf{Id}}[\overline{u}(\psi)]\colon u^*\psi \to u^*\psi}\, F_4$$

A functor p satisfies F_5 if for arrow $v\colon \Lambda \to \Gamma$, $u\colon \Gamma \to \Delta$ in B and arrow f in E such that $pf = u \circ v$, the arrow $\overline{u}(\psi) \circ u_v[f]$ is equal to the arrow f.

$$\frac{v\colon \Lambda \to \Gamma \quad u\colon \Gamma \to \Delta \quad u \circ v\colon \Lambda \to \Delta \vdash f\colon \varphi \to \psi}{u \circ v\colon \Lambda \to \Delta \vdash f = \overline{u}(\psi) \circ u_v[f]\colon \varphi \to \psi}\, F_5$$

A functor p satisfies F_6 if for arrow $w\colon \Xi \to \Lambda$, $v\colon \Lambda \to \Gamma$, $u\colon \Gamma \to \Delta$ in B and arrow $g\colon \sigma \to \varphi$ and $f\colon \varphi \to \psi$ in E such that $pg = w$ and $pf = u \circ v$, the arrow $u_v[f] \circ g$ is equal to the arrow $u_{v \circ w}[f \circ g]$.

$$\frac{v\colon \Lambda \to \Gamma \quad u\colon \Gamma \to \Delta \quad u \circ v\colon \Lambda \to \Delta \vdash f\colon \varphi \to \psi \quad w\colon \Xi \to \Lambda \vdash g\colon \sigma \to \varphi}{v \circ w\colon \Xi \to \Gamma \vdash u_{v \circ w}[f \circ g] = u_v[f] \circ g\colon \sigma \to u^*\psi}\, F_6$$

Theorem 4.6.1. A functor $p\colon E \to B$ satisfies properties F_1 through F_6 if and only if p is a cloven fibration.

Proof. Here, we prove only that if a functor p satisfies the properties F_1 through F_6 then $\overline{u}(\psi)$ is cartesian over u. By F_3 and F_5, for arrow $v\colon \Lambda \to \Gamma$, $u\colon \Gamma \to \Delta$ in B and arrow $f\colon \varphi \to \psi$ in E such that $pf = u \circ v$, there exists an arrow $u_v[f]\colon \varphi \to u^*\psi$ in E such that $p(u_v[f]) = v$ and $f = \overline{u}(\psi) \circ u_v[f]$. Let $g\colon \varphi \to u^*\psi$ satisfy $pg = v$ and $f = \overline{u}(\psi) \circ g$. By F_6, $u_{\mathbf{Id}}[\overline{u}(\psi)] \circ g$ is equal to $u_v[\overline{u}(\psi) \circ g]$. By $f = \overline{u}(\psi) \circ g$ and F_4, g is equal to $u_v[f]$. Therefore, $u_v[f]$ is a unique arrow such that $p(u_v[f]) = v$ and $f = \overline{u}(\psi) \circ u_v[f]$. $\square$

4.6.2 Lawvere $\mathbf{Cat}_o^{\rightarrow}$-Theory Fb for Cloven Fibrations

Putting $V = \mathbf{Set}$ and $A = \mathbf{Cat}_o^{\rightarrow}$, we give Lawvere $\mathbf{Cat}_o^{\rightarrow}$-theories for which the models are all cloven fibrations. We can check that $\mathbf{Cat}_o^{\rightarrow}$ is a locally finitely presentable category.

For later use, we define the category **4** and name the arrows of the category **3**.

- **4** is the category freely generated from the following graph.

$$\cdot \xrightarrow{\textbf{first}} \cdot \xrightarrow{\textbf{second}} \cdot \xrightarrow{\textbf{third}} \cdot$$

- **3** is the category freely generated from the following graph.

$$\cdot \xrightarrow{\textbf{pre}} \cdot \xrightarrow{\textbf{post}} \cdot$$

For later use, we define some finitely presentable objects in $\mathbf{Cat}_o^{\rightarrow}$.

- $?_1$ is the unique functor from 0 to 1.
- $?_{\mathbf{2}}$ is the unique functor from 0 to **2**.
- $\mathbf{Id}_1$, $\mathbf{Id}_{\mathbf{2}}$, and $\mathbf{Id}_{\mathbf{3}}$ are the identity functors on 1, **2**, and **3**, respectively.
- **cod** is a functor from the category 1 to the category **2**, which sends the unique object of 1 to the codomain of the non-identity arrow in **2**.

- **comp** is a functor from the category **2** to the category **3**, which sends the non-identity arrow of **2** to **post** $\circ$ **pre**.
- **ext** is a functor from the category **3** to the category **4**, which sends **pre** to **first** and sends **post** to **third** $\circ$ **second**.

Let **Fb0** be the freely generated Lawvere $\mathbf{Cat}_o^{\to}$-theory from $(\mathbf{Cat}_o^{\to})_f^{\mathrm{op}}$ by adding the new arrows called as follows.

$$\begin{array}{ll} \mathbf{F}_1\colon \mathbf{cod} \to \mathbf{Id}_1 & \mathbf{F}_4\colon \mathbf{cod} \to \mathbf{Id_2} \\ \mathbf{F}_2\colon \mathbf{cod} \to \mathbf{Id_2} & \mathbf{F}_5\colon \mathbf{comp} \to \mathbf{Id_3} \\ \mathbf{F}_3\colon \mathbf{comp} \to \mathbf{Id_2} & \mathbf{F}_6\colon \mathbf{ext} \to \mathbf{Id_3} \end{array}$$

Next, we give certain equations among arrows of the Lawvere $\mathbf{Cat}_o^{\to}$-theory **Fb0**. Let (p, S) be a model of **Fb0**.

There exists $\mathbf{H}_1\colon \mathbf{cod} \to ?_\mathbf{2}$ such that for any arrow $u\colon \Gamma \to \Delta$ in B and object ψ in E such that $p\psi = \Delta$, the function $S\mathbf{H}_1$ returns u.

$$\frac{u\colon \Gamma \to \Delta \quad \Delta \vdash \psi}{u\colon \Gamma \to \Delta}\ \mathbf{H}_1$$

There exists $\mathbf{H}_2\colon ?_\mathbf{2} \to ?_1$ such that for any arrow $u\colon \Gamma \to \Delta$ in B, the function $S\mathbf{H}_2$ returns Γ.

$$\frac{u\colon \Gamma \to \Delta}{\Gamma\colon \mathbf{base}}\ \mathbf{H}_2$$

There exists $\mathbf{H}_3\colon \mathbf{Id}_1 \to ?_1$ such that for any object Δ in B and object ψ in E such that $p\psi = \Delta$, the function $S\mathbf{H}_3$ returns Δ.

$$\frac{\Delta \vdash \psi}{\Delta\colon \mathbf{base}}\ \mathbf{H}_3$$

There exists $\mathbf{H}_4\colon \mathbf{cod} \to \mathbf{Id}_1$ such that for any arrow $u\colon \Gamma \to \Delta$ in B and object ψ in E such that $p\psi = \Delta$, the function $S\mathbf{H}_4$ returns Δ and ψ.

$$\frac{u\colon \Gamma \to \Delta \quad \Delta \vdash \psi}{\Delta \vdash \psi}\ \mathbf{H}_4$$

There exists $\mathbf{H}_5\colon \mathbf{Id_2} \to \mathbf{Id}_1$ such that for any arrow $u\colon \Gamma \to \Delta$ in B and arrow $f\colon \varphi \to \psi$ in E such that $pf = u$, the function $S\mathbf{H}_5$ returns Δ and ψ.

$$\frac{u\colon \Gamma \to \Delta \vdash f\colon \varphi \to \psi}{\Delta \vdash \psi}\ \mathbf{H}_5$$

There exists $\mathbf{H}_6\colon \mathbf{Id_2} \to ?_\mathbf{2}$ such that for any arrow $u\colon \Gamma \to \Delta$ in B and arrow $f\colon \varphi \to \psi$ in E such that $pf = u$, the function $S\mathbf{H}_6$ returns u.

$$\frac{u\colon \Gamma \to \Delta \vdash f\colon \varphi \to \psi}{u\colon \Gamma \to \Delta}\ \mathbf{H}_6$$

There exists $\mathbf{H}_7\colon \mathbf{comp} \to \mathbf{Id}_\to$ such that for any arrow $v\colon \Lambda \to \Gamma$, $u\colon \Gamma \to \Delta$ in B and arrow $f\colon \varphi \to \psi$ in E such that $pf = u \circ v$, the function $S\mathbf{H}_7$ returns f and $u \circ v$.

$$\frac{v\colon \Lambda \to \Gamma \quad u\colon \Gamma \to \Delta \quad u \circ v\colon \Lambda \to \Delta \vdash f\colon \varphi \to \psi}{u \circ v\colon \Lambda \to \Delta \vdash f\colon \varphi \to \psi}\ \mathbf{H}_7$$

There exists $\mathbf{H}_8\colon \mathbf{Id}_\to \to \mathbf{Id}_1$ such that for any arrow $u\colon \Gamma \to \Delta$ in B and arrow $f\colon \varphi \to \psi$ in E such that $pf = u$, the function $S\mathbf{H}_8$ returns Γ and φ.

$$\frac{u\colon \Gamma \to \Delta \vdash f\colon \varphi \to \psi}{\Gamma \vdash \varphi}\ \mathbf{H}_8$$

There exists $\mathbf{H}_9\colon \mathbf{comp} \to \mathbf{cod}$ such that for any arrow $v\colon \Lambda \to \Gamma$, $u\colon \Gamma \to \Delta$ in B and arrow $f\colon \varphi \to \psi$ in E such that $pf = u \circ v$, the function $S\mathbf{H}_9$ returns u and ψ.

$$\frac{v\colon \Lambda \to \Gamma \quad u\colon \Gamma \to \Delta \quad u \circ v\colon \Lambda \to \Delta \vdash f\colon \varphi \to \psi}{u\colon \Gamma \to \Delta \quad \Delta \vdash \psi}\ \mathbf{H}_9$$

There exists $\mathbf{H}_{10}\colon \mathbf{Id}_1 \to \mathbf{Id_2}$ such that for any object Δ in B and object ψ in E such that $p\psi = \Delta$, the function $S\mathbf{H}_{10}$ returns $\mathbf{Id}\colon \Delta \to \Delta$ and $\mathbf{Id}\colon \psi \to \psi$.

$$\frac{\Delta \vdash \psi}{\mathbf{Id}\colon \Delta \to \Delta \vdash \mathbf{Id}\colon \psi \to \psi}\ \mathbf{H}_{10}$$

There exists $\mathbf{H}_{11}\colon \mathbf{Id_2} \to \mathbf{comp}$ such that for any arrow $u\colon \Gamma \to \Delta$ in B and arrow $f\colon \varphi \to \psi$ in E such that $pf = u$, the function $S\mathbf{H}_{11}$ returns u, f, and $\mathbf{Id}\colon \Gamma \to \Gamma$.

$$\frac{u\colon \Gamma \to \Delta \vdash f\colon \varphi \to \psi}{\mathbf{Id}\colon \Gamma \to \Gamma \quad u\colon \Gamma \to \Delta \quad u \circ \mathbf{Id}\colon \Gamma \to \Delta \vdash f\colon \varphi \to \psi}\ \mathbf{H}_{11}$$

There exists $\mathbf{H}_{12}\colon \mathbf{Id_3} \to \mathbf{Id_2}$ such that for any arrow $v\colon \Lambda \to \Gamma$, $u\colon \Gamma \to \Delta$ in B and arrow $g\colon \sigma \to \varphi$, $f\colon \varphi \to \psi$ in E such that $pg = v$ and $pf = u$, the function $S\mathbf{H}_{12}$ returns $f \circ g$ and $u \circ v$.

$$\frac{v\colon \Lambda \to \Gamma \vdash g\colon \sigma \to \varphi \quad u\colon \Gamma \to \Delta \vdash f\colon \varphi \to \psi}{u \circ v\colon \Lambda \to \Delta \vdash f \circ g\colon \sigma \to \psi}\ \mathbf{H}_{12}$$

There exists $\mathbf{H}_{13}\colon \mathbf{Id_3} \to \mathbf{Id_2}$ such that for any arrow $v\colon \Lambda \to \Gamma$, $u\colon \Gamma \to \Delta$ in B and arrow $g\colon \sigma \to \varphi$, $f\colon \varphi \to \psi$ in E such that $pg = v$ and $pf = u$, the function $S\mathbf{H}_{13}$ returns f and u.

$$\frac{v\colon \Lambda \to \Gamma \vdash g\colon \sigma \to \varphi \quad u\colon \Gamma \to \Delta \vdash f\colon \varphi \to \psi}{u\colon \Gamma \to \Delta \vdash f\colon \varphi \to \psi}\ \mathbf{H}_{13}$$

There exists $\mathbf{H}_{14}\colon \mathbf{Id_3} \to \mathbf{Id_2}$ such that for any arrow $v\colon \Lambda \to \Gamma$, $u\colon \Gamma \to \Delta$ in B and arrow $g\colon \sigma \to \varphi$, $f\colon \varphi \to \psi$ in E such that $pg = v$ and $pf = u$, the function $S\mathbf{H}_{14}$ returns g and v.

$$\frac{v\colon \Lambda \to \Gamma \vdash g\colon \sigma \to \varphi \quad u\colon \Gamma \to \Delta \vdash f\colon \varphi \to \psi}{v\colon \Lambda \to \Gamma \vdash g\colon \sigma \to \varphi}\ \mathbf{H}_{14}$$

There exists $\mathbf{H}_{15}\colon \mathbf{ext} \to \mathbf{Id_3}$ such that for any arrow $w\colon \Xi \to \Lambda$, $v\colon \Lambda \to \Gamma$, $u\colon \Gamma \to \Delta$ in B and arrow $g\colon \sigma \to \varphi$ and $f\colon \varphi \to \psi$ in E such that $pg = w$ and $pf = u \circ v$, the function $S\mathbf{H}_{15}$ returns w, $u \circ v$, g, and f.

$$\frac{v\colon \Lambda \to \Gamma \quad u\colon \Gamma \to \Delta \quad u \circ v\colon \Lambda \to \Delta \vdash f\colon \varphi \to \psi \quad w\colon \Xi \to \Lambda \vdash g\colon \sigma \to \varphi}{u \circ v\colon \Lambda \to \Delta \vdash f\colon \varphi \to \psi \quad w\colon \Xi \to \Lambda \vdash g\colon \sigma \to \varphi}\ \mathbf{H}_{15}$$

There exists $\mathbf{H}_{16}\colon \mathbf{ext} \to \mathbf{comp}$ such that for any arrow $w\colon \Xi \to \Lambda$, $v\colon \Lambda \to \Gamma$, $u\colon \Gamma \to \Delta$ in B and arrow $g\colon \sigma \to \varphi$ and $f\colon \varphi \to \psi$ in E such that $pg = w$ and $pf = u \circ v$, the function $S\mathbf{H}_{16}$ returns u, v, and f.

$$\frac{v\colon \Lambda \to \Gamma \quad u\colon \Gamma \to \Delta \quad u \circ v\colon \Lambda \to \Delta \vdash f\colon \varphi \to \psi \quad w\colon \Xi \to \Lambda \vdash g\colon \sigma \to \varphi}{v\colon \Lambda \to \Gamma \quad u\colon \Gamma \to \Delta \quad u \circ v\colon \Lambda \to \Delta \vdash f\colon \varphi \to \psi}\ \mathbf{H}_{16}$$

There exists $\mathbf{H}_{17}\colon \mathbf{ext} \to \mathbf{comp}$ such that for any arrow $w\colon \Xi \to \Lambda$, $v\colon \Lambda \to \Gamma$, $u\colon \Gamma \to \Delta$ in B and arrow $g\colon \sigma \to \varphi$ and $f\colon \varphi \to \psi$ in E such that $pg = w$ and $pf = u \circ v$, the function $S\mathbf{H}_{17}$ returns u, $v \circ w$, and $f \circ g$.

$$\frac{v\colon \Lambda \to \Gamma \quad u\colon \Gamma \to \Delta \quad u \circ v\colon \Lambda \to \Delta \vdash f\colon \varphi \to \psi \quad w\colon \Xi \to \Lambda \vdash g\colon \sigma \to \varphi}{v \circ w\colon \Xi \to \Gamma \quad u\colon \Gamma \to \Delta \quad u \circ v \circ w\colon \Xi \to \Delta \vdash f \circ g\colon \sigma \to \psi}\ \mathbf{H}_{17}$$

There exists $\mathbf{H}_{18}\colon \mathbf{comp} \to ?_{\mathbf{2}}$ such that for any arrow $v\colon \Lambda \to \Gamma$, $u\colon \Gamma \to \Delta$ in B and arrow $f\colon \varphi \to \psi$ in E such that $pf = u \circ v$, the function $S\mathbf{H}_{18}$ returns v.

$$\frac{v\colon \Lambda \to \Gamma \quad u\colon \Gamma \to \Delta \quad u \circ v\colon \Lambda \to \Delta \vdash f\colon \varphi \to \psi}{v\colon \Lambda \to \Gamma}\ \mathbf{H}_{18}$$

Let **Fb** be the freely generated Lawvere $\mathbf{Cat}_o^{\rightarrow}$-theory from **Fb0** subject to the following equations.

$$\begin{array}{rcl} \mathbf{H}_3 \circ \mathbf{F}_1 &=& \mathbf{H}_2 \circ \mathbf{H}_1 \\ \mathbf{H}_5 \circ \mathbf{F}_2 &=& \mathbf{H}_4 \\ \mathbf{H}_6 \circ \mathbf{F}_2 &=& \mathbf{H}_1 \\ \mathbf{H}_8 \circ \mathbf{F}_2 &=& \mathbf{F}_1 \\ \mathbf{H}_5 \circ \mathbf{F}_3 &=& \mathbf{F}_1 \circ \mathbf{H}_9 \\ \mathbf{H}_6 \circ \mathbf{F}_3 &=& \mathbf{H}_{18} \\ \mathbf{H}_8 \circ \mathbf{F}_3 &=& \mathbf{H}_8 \circ \mathbf{H}_7 \end{array} \qquad \begin{array}{rcl} \mathbf{F}_4 &=& \mathbf{H}_{10} \circ \mathbf{F}_1 \\ \mathbf{F}_4 &=& \mathbf{F}_3 \circ \mathbf{H}_{11} \circ \mathbf{F}_2 \\ \mathbf{H}_{12} \circ \mathbf{F}_5 &=& \mathbf{H}_7 \\ \mathbf{H}_{13} \circ \mathbf{F}_5 &=& \mathbf{F}_2 \circ \mathbf{H}_9 \\ \mathbf{H}_{14} \circ \mathbf{F}_5 &=& \mathbf{F}_3 \\ \mathbf{H}_{12} \circ \mathbf{F}_6 &=& \mathbf{F}_3 \circ \mathbf{H}_{17} \\ \mathbf{H}_{13} \circ \mathbf{F}_6 &=& \mathbf{F}_3 \circ \mathbf{H}_{16} \\ \mathbf{H}_{14} \circ \mathbf{F}_6 &=& \mathbf{H}_{14} \circ \mathbf{H}_{15} \end{array}$$

Theorem 4.6.2. There exists a Lawvere $\mathbf{Cat}_o^{\rightarrow}$-theory **Fb** for which the models are all cloven fibrations.

4.6.3 Lawvere $\mathbf{Cat}_o^{\rightarrow}$-Theories for Fibrations with Structures

Putting $V = \mathbf{Set}$ and $A = \mathbf{Cat}_o^{\rightarrow}$, we give Lawvere $\mathbf{Cat}_o^{\rightarrow}$-theories for which the models are all cloven fibrations with a fibred terminal object, all cloven fibrations with fibred finite limits, all cloven fibrations with fibred finite colimits, all cocomplete fibrations, all complete fibrations, and all fibred CCCs, respectively [35, 36]. For space reasons, we show only rule-based presentation for fibred structures.

There exists a Lawvere $\mathbf{Cat}_o^{\rightarrow}$-theory **Fbt** for which the models are all cloven fibrations that have a fibred terminal object. In the following rules, the object 1_Γ represents a terminal object of a fibre on Γ. The arrow $!_\psi$ represents a canonical arrow from ψ to the terminal object. The rules T_3 and T_4 imply uniqueness of the arrow. The arrow $!^{-1}_{u^*1_\Delta}$ represents the inverse arrow of $!_{u^*1_\Delta}$.

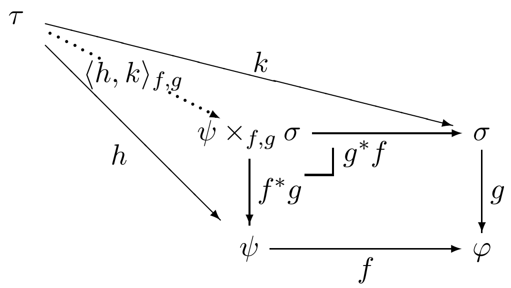
$$\frac{\Gamma\colon \mathbf{base}}{\Gamma \vdash 1_\Gamma}\,T_1 \qquad \frac{\Gamma \vdash \psi}{\Gamma \vdash !_\psi\colon \psi \to 1_\Gamma}\,T_2$$

$$\frac{\Gamma\colon \mathbf{base}}{\Gamma \vdash \mathbf{Id} = !_{1_\Gamma}\colon 1_\Gamma \to 1_\Gamma}\,T_3 \qquad \frac{\Gamma \vdash f\colon \psi \to \varphi}{\Gamma \vdash !_\varphi \circ f = !_\psi\colon \psi \to 1_\Gamma}\,T_4$$

$$\frac{u\colon \Gamma \to \Delta}{\Gamma \vdash !^{-1}_{u^*1_\Delta}\colon 1_\Gamma \to u^*1_\Delta}\,T_5 \qquad \frac{u\colon \Gamma \to \Delta}{\Gamma \vdash \mathbf{Id} = !^{-1}_{u^*1_\Delta} \circ !_{u^*1_\Delta}\colon u^*1_\Delta \to u^*1_\Delta}\,T_6$$

There exists a Lawvere $\mathbf{Cat}_o^{\rightarrow}$-theory **Fbfl** for which the models are all cloven fibrations that have fibred finite limits. The object $\psi \times_{f,g} \sigma$ and arrows f^*g and

g^*f represent the following pullback.



The rules P_1 through P_4 represent that the pullback diagram commutes.

$$\frac{\Gamma \vdash f\colon \psi \to \varphi \quad \Gamma \vdash g\colon \sigma \to \varphi}{\Gamma \vdash \psi \times_{f,g} \sigma}\ P_1 \qquad \frac{\Gamma \vdash f\colon \psi \to \varphi \quad \Gamma \vdash g\colon \sigma \to \varphi}{\Gamma \vdash f^*g\colon \psi \times_{f,g} \sigma \to \psi}\ P_2$$

$$\frac{\Gamma \vdash f\colon \psi \to \varphi \quad \Gamma \vdash g\colon \sigma \to \varphi}{\Gamma \vdash g^*f\colon \psi \times_{f,g} \sigma \to \sigma}\ P_3 \qquad \frac{\Gamma \vdash f\colon \psi \to \varphi \quad \Gamma \vdash g\colon \sigma \to \varphi}{\Gamma \vdash f \circ f^*g = g \circ g^*f\colon \psi \times_{f,g} \sigma \to \varphi}\ P_4$$

The rules P_5 through P_7 represent existence of canonical arrow to the pullback. The rules P_8 and P_9 imply uniqueness of the arrow.

$$\frac{\Gamma \vdash f\colon \psi \to \varphi \quad \Gamma \vdash g\colon \sigma \to \varphi \quad \Gamma \vdash f \circ h = g \circ k\colon \tau \to \varphi}{\Gamma \vdash \langle h, k\rangle_{f,g}\colon \tau \to \psi \times_{f,g} \sigma}\ P_5$$

$$\frac{\Gamma \vdash f\colon \psi \to \varphi \quad \Gamma \vdash g\colon \sigma \to \varphi \quad \Gamma \vdash f \circ h = g \circ k\colon \tau \to \varphi}{\Gamma \vdash f^*g \circ \langle h, k\rangle_{f,g} = h\colon \tau \to \psi}\ P_6$$

$$\frac{\Gamma \vdash f\colon \psi \to \varphi \quad \Gamma \vdash g\colon \sigma \to \varphi \quad \Gamma \vdash f \circ h = g \circ k\colon \tau \to \varphi}{\Gamma \vdash g^*f \circ \langle h, k\rangle_{f,g} = k\colon \tau \to \sigma}\ P_7$$

$$\frac{\Gamma \vdash f\colon \psi \to \varphi \quad \Gamma \vdash g\colon \sigma \to \varphi}{\Gamma \vdash \mathbf{Id} = \langle f^*g, g^*f\rangle_{f,g}\colon \psi \times_{f,g} \sigma \to \psi \times_{f,g} \sigma}\ P_8$$

$$\frac{\Gamma \vdash f\colon \psi \to \varphi \quad \Gamma \vdash g\colon \sigma \to \varphi \quad \Gamma \vdash f \circ h = g \circ k\colon \tau \to \varphi \quad \Gamma \vdash l\colon \rho \to \tau}{\Gamma \vdash \langle h \circ l, k \circ l\rangle_{f,g} = \langle h, k\rangle_{f,g} \circ l\colon \rho \to \psi \times_{f,g} \sigma}\ P_9$$

In the rule P_{10} and P_{12}, we write u^*f for $u_{\mathbf{Id}}[f \circ \overline{u}(\varphi)]\colon u^*\varphi \to u^*\psi$. It represents the arrow part of the reindexing functor u^*. In the rule P_{11} and P_{12}, we use the following abbreviation $\alpha(u, f, g)$. It represents canonical isomorphism for pullbacks preserved by reindexing functors.

$$\alpha(u, f, g) = \langle u^*(f^*g), u^*(g^*f)\rangle_{u^*f, u^*g}$$

$$\frac{u\colon \Gamma \to \Delta \quad \Delta \vdash f\colon \psi \to \varphi \quad \Delta \vdash g\colon \sigma \to \varphi}{\Gamma \vdash \alpha(u, f, g)^{-1}\colon (u^*\psi) \times_{u^*f, u^*g} (u^*\sigma) \to u^*(\psi \times_{f,g} \sigma)}\ P_{10}$$

$$\frac{u\colon \Gamma \to \Delta \quad \Delta \vdash f\colon \psi \to \varphi \quad \Delta \vdash g\colon \sigma \to \varphi}{\Gamma \vdash \mathbf{Id} = \alpha(u,f,g)^{-1} \circ \alpha(u,f,g)\colon u^*(\psi \times_{f,g} \sigma) \to u^*(\psi \times_{f,g} \sigma)}\ P_{11}$$

$$\frac{u\colon \Gamma \to \Delta \quad \Delta \vdash f\colon \psi \to \varphi \quad \Delta \vdash g\colon \sigma \to \varphi}{\Gamma \vdash \mathbf{Id} = \alpha(u,f,g) \circ \alpha(u,f,g)^{-1}\colon (u^*\psi) \times_{u^*f,u^*g} (u^*\sigma) \to (u^*\psi) \times_{u^*f,u^*g} (u^*\sigma)}\ P_{12}$$

Similarly, there exists a Lawvere $\mathbf{Cat}_o^{\to}$-theory **Fbfc** for which the models are all cloven fibrations that have fibred finite colimits.

There exists a Lawvere $\mathbf{Cat}_o^{\to}$-theory **Fbco** for which the models are all cocomplete fibrations. The rules L_1 and L_2 represent the object part and the arrow part of the left adjoint Σ_u of u^*, respectively. The rules L_1 through L_4 represent that Σ_u is a functor.

$$\frac{u\colon \Gamma \to \Delta \quad \Gamma \vdash \psi}{\Delta \vdash \Sigma_u(\psi)}\ L_1 \qquad \frac{u\colon \Gamma \to \Delta \quad \Gamma \vdash f\colon \psi \to \varphi}{\Delta \vdash \Sigma_u(f)\colon \Sigma_u(\psi) \to \Sigma_u(\varphi)}\ L_2$$

$$\frac{u\colon \Gamma \to \Delta \quad \Gamma \vdash \psi}{\Delta \vdash \mathbf{Id} = \Sigma_u(\mathbf{Id})\colon \Sigma_u(\psi) \to \Sigma_u(\psi)}\ L_3$$

$$\frac{u\colon \Gamma \to \Delta \quad \Gamma \vdash f\colon \psi \to \varphi \quad \Gamma \vdash g\colon \varphi \to \sigma}{\Delta \vdash \Sigma_u(g \circ f) = \Sigma_u(g) \circ \Sigma_u(f)\colon \Sigma_u(\psi) \to \Sigma_u(\sigma)}\ L_4$$

The rule U_1 represents the φ-component $\eta_u(\varphi)$ of the unit η_u of the adjunction $\Sigma_u \dashv u^*$. The rule C_1 represents the φ-component $\epsilon_u(\varphi)$ of the counit ϵ_u of the adjunction. The rules U_2 and C_2 represent naturality of them. The rules A_1 and A_2 represent triangle conditions for the adjunction $\Sigma_u \dashv u^*$.

$$\frac{u\colon \Gamma \to \Delta \quad \Gamma \vdash \varphi}{\Gamma \vdash \eta_u(\varphi)\colon \varphi \to u^*(\Sigma_u(\varphi))}\ U_1 \qquad \frac{u\colon \Gamma \to \Delta \quad \Delta \vdash \varphi}{\Delta \vdash \epsilon_u(\varphi)\colon \Sigma_u(u^*(\varphi)) \to \varphi}\ C_1$$

$$\frac{u\colon \Gamma \to \Delta \quad \Gamma \vdash f\colon \varphi \to \psi}{\Gamma \vdash u^*(\Sigma_u(f)) \circ \eta_u(\varphi) = \eta_u(\psi) \circ f\colon \varphi \to u^*(\Sigma_u(\psi))}\ U_2$$

$$\frac{u\colon \Gamma \to \Delta \quad \Delta \vdash f\colon \varphi \to \psi}{\Delta \vdash f \circ \epsilon_u(\varphi) = \epsilon_u(\psi) \circ \Sigma_u(u^*(f))\colon \Sigma_u(u^*(\varphi)) \to \psi}\ C_2$$

$$\frac{u\colon \Gamma \to \Delta \quad \Gamma \vdash \varphi}{\Delta \vdash \mathbf{Id} = \epsilon_u(\Sigma_u(\varphi)) \circ \Sigma_u(\eta_u(\varphi))\colon \Sigma_u(\varphi) \to \Sigma_u(\varphi)}\ A_1$$

$$\frac{u\colon \Gamma \to \Delta \quad \Delta \vdash \varphi}{\Gamma \vdash \mathbf{Id} = u^*(\epsilon_u(\varphi)) \circ \eta_u(u^*(\varphi))\colon u^*(\varphi) \to u^*(\varphi)}\ A_2$$

Similarly to fibred finite limits, we can define a Lawvere $\mathbf{Cat}_o^{\to}$-theory for which the models are all cloven fibrations $p\colon \mathbf{E} \to \mathbf{B}$ that have pullbacks in the base category **B**. Therefore, we write the pullback of u and x as follows.

$$\begin{array}{ccc} \Xi \times_{x,u} \Gamma & \xrightarrow{u^* x} & \Gamma \\ {\scriptstyle x^* u}\downarrow & & \downarrow{\scriptstyle u} \\ \Xi & \xrightarrow[x]{} & \Delta \end{array}$$

In the rules B_2 and B_3, we use the following abbreviations $\beta(u, x, \psi)$. It represents ψ-component of the canonical natural transformation induced by the pullback of u and x.

$$\delta_{u,x}(\Sigma_u(\psi)) = (x^*u)_{\mathbf{Id}}[x_{x^*u}[\overline{u}(\Sigma_u(\psi)) \circ \overline{u^*x}(u^*(\Sigma_u(\psi)))]]$$

$$\beta(u, x, \psi) = \epsilon_{x^*u}(x^*(\Sigma_u(\psi))) \circ \Sigma_{x^*u}(\delta_{u,x}(\Sigma_u(\psi)) \circ (u^*x)(\eta_u(\psi)))$$

$$\frac{u\colon \Gamma \to \Delta \quad x\colon \Xi \to \Delta \quad \Gamma \vdash \psi}{\Xi \vdash \beta(u, x, \psi)^{-1}\colon\ x^*(\Sigma_u(\psi)) \to \Sigma_{x^*u}((u^*x)^*(\psi))}\ B_1$$

$$\frac{u\colon \Gamma \to \Delta \quad x\colon \Xi \to \Delta \quad \Gamma \vdash \psi}{\Xi \vdash \mathbf{Id} = \beta(u, x, \psi)^{-1} \circ \beta(u, x, \psi)\colon\ \Sigma_{x^*u}((u^*x)^*(\psi)) \to \Sigma_{x^*u}((u^*x)^*(\psi))}\ B_2$$

$$\frac{u\colon \Gamma \to \Delta \quad x\colon \Xi \to \Delta \quad \Gamma \vdash \psi}{\Xi \vdash \mathbf{Id} = \beta(u, x, \psi) \circ \beta(u, x, \psi)^{-1}\colon\ x^*(\Sigma_u(\psi)) \to x^*(\Sigma_u(\psi))}\ B_3$$

Similarly, there exists a Lawvere $\mathbf{Cat}_o^{\rightarrow}$-theory **Fbc** for which the models are all complete fibrations.

There exists a Lawvere $\mathbf{Cat}_o^{\rightarrow}$-theory **Fbccc** for which the models are all fibred CCCs. Since fibred finite products are fibred finite limits, we can define a Lawvere $\mathbf{Cat}_o^{\rightarrow}$-theory for which the models are all cloven fibrations that have fibred finite products. For space reasons, we show only four rules for fibred binary products. The rules E_1 and E_2 represent the object part and the arrow part of the endofunctor $(-) \times \sigma$ on the fibre $\mathbf{E}_\Gamma$ for each object σ of $\mathbf{E}_\Gamma$. The rules E_3 and E_4 represent the isomorphisms for binary products preserved by reindexing functors.

$$\frac{\Gamma \vdash \psi \quad \Gamma \vdash \sigma}{\Gamma \vdash \psi \times \sigma}\ E_1 \qquad \frac{\Gamma \vdash f\colon \psi \to \varphi \quad \Gamma \vdash \sigma}{\Gamma \vdash f \times \sigma\colon\ \psi \times \sigma \to \varphi \times \sigma}\ E_2$$

$$\frac{u\colon \Gamma \to \Delta \quad \Delta \vdash \sigma \quad \Delta \vdash \psi}{\Gamma \vdash \mu(u, \psi, \sigma)\colon\ u^*(\psi \times \sigma) \to u^*\psi \times u^*\sigma}\ E_3$$

$$\frac{u\colon \Gamma \to \Delta \quad \Delta \vdash \sigma \quad \Delta \vdash \psi}{\Gamma \vdash \mu(u, \psi, \sigma)^{-1}\colon\ u^*\psi \times u^*\sigma \to u^*(\psi \times \sigma)}\ E_4$$

The rules E_5 and E_6 represent the object part and the arrow part of the right adjoint $[\sigma, -]$ of $(-) \times \sigma$, respectively. The rules E_5 through E_8 represent that $[\sigma, -]$ is a functor.

$$\frac{\Gamma \vdash \sigma \quad \Gamma \vdash \psi}{\Gamma \vdash [\sigma, \psi]} E_5 \quad \frac{\Gamma \vdash \sigma \quad \Gamma \vdash f\colon \psi \to \varphi}{\Gamma \vdash [\sigma, f]\colon [\sigma, \psi] \to [\sigma, \varphi]} E_6$$

$$\frac{\Gamma \vdash \sigma \quad \Gamma \vdash \psi}{\Gamma \vdash \mathbf{Id} = [\sigma, \mathbf{Id}]\colon [\sigma, \psi] \to [\sigma, \psi]} E_7$$

$$\frac{\Gamma \vdash \sigma \quad \Gamma \vdash f\colon \psi \to \varphi \quad \Gamma \vdash g\colon \varphi \to \tau}{\Gamma \vdash [\sigma, g] \circ [\sigma, f] = [\sigma, g \circ f]\colon [\sigma, \psi] \to [\sigma, \tau]} E_8$$

The rule E_9 represents the ψ-component $\eta_\sigma(\psi)$ of the unit η_σ of the adjunction $(-) \times \sigma \dashv [\sigma, -]$. The rule E_{10} represents the ψ-component $\epsilon_\sigma(\psi)$ of the counit ϵ_σ of the adjunction. The rules E_{11} and E_{12} represent naturality of them. The rules E_{13} and E_{14} represent triangle conditions for the adjunction $(-) \times \sigma \dashv [\sigma, -]$.

$$\frac{\Gamma \vdash \sigma \quad \Gamma \vdash \psi}{\Gamma \vdash \eta_\sigma(\psi)\colon \psi \to [\sigma, \psi \times \sigma]} E_9 \quad \frac{\Gamma \vdash \sigma \quad \Gamma \vdash \psi}{\Gamma \vdash \epsilon_\sigma(\psi)\colon [\sigma, \psi] \times \sigma \to \psi} E_{10}$$

$$\frac{\Gamma \vdash \sigma \quad \Gamma \vdash f\colon \psi \to \varphi}{\Gamma \vdash \eta_\sigma(\varphi) \circ f = [\sigma, f \times \sigma] \circ \eta_\sigma(\psi)\colon \psi \to [\sigma, \varphi \times \sigma]} E_{11}$$

$$\frac{\Gamma \vdash \sigma \quad \Gamma \vdash f\colon \psi \to \varphi}{\Gamma \vdash f \circ \epsilon_\sigma(\psi) = \epsilon_\sigma(\varphi) \circ ([\sigma, f] \times \sigma)\colon [\sigma, \psi] \times \sigma \to \varphi} E_{12}$$

$$\frac{\Gamma \vdash \sigma \quad \Gamma \vdash \psi}{\Gamma \vdash \mathbf{Id} = [\sigma, \epsilon_\sigma(\psi)] \circ \eta_\sigma([\sigma, \psi])\colon [\sigma, \psi] \to [\sigma, \psi]} E_{13}$$

$$\frac{\Gamma \vdash \sigma \quad \Gamma \vdash \psi}{\Gamma \vdash \mathbf{Id} = \epsilon_\sigma(\psi \times \sigma) \circ (\eta_\sigma(\psi) \times \sigma)\colon \psi \times \sigma \to \psi \times \sigma} E_{14}$$

In the rule E_{16} and E_{17}, we use the following abbreviation $\nu(u, \sigma, \psi)$. It represents canonical isomorphism for exponents preserved by reindexing functors.

$$\nu(u, \sigma, \psi) = [u^*\sigma, u^*\epsilon_\sigma(\psi)] \circ [u^*\sigma, \mu(u, [\sigma, \psi], \sigma)^{-1}] \circ \eta_{u^*\sigma}(u^*[\sigma, \psi])$$

$$\frac{u\colon \Gamma \to \Delta \quad \Delta \vdash \sigma \quad \Delta \vdash \psi}{\Gamma \vdash \nu(u, \sigma, \psi)^{-1}\colon [u^*\sigma, u^*\psi] \to u^*[\sigma, \psi]} E_{15}$$

$$\frac{u\colon \Gamma \to \Delta \quad \Delta \vdash \sigma \quad \Delta \vdash \psi}{\Gamma \vdash \mathbf{Id} = \nu(u, \sigma, \psi)^{-1} \circ \nu(u, \sigma, \psi)\colon u^*[\sigma, \psi] \to u^*[\sigma, \psi]} E_{16}$$

$$\frac{u\colon \Gamma \to \Delta \quad \Delta \vdash \sigma \quad \Delta \vdash \psi}{\Gamma \vdash \mathbf{Id} = \nu(u, \sigma, \psi) \circ \nu(u, \sigma, \psi)^{-1}\colon [u^*\sigma, u^*\psi] \to [u^*\sigma, u^*\psi]} E_{17}$$

By Definition 3.3.1, there exists a Lawvere $\mathbf{Cat}_o^{\to}$-theory **GLTS** for which the models are all **GLTS**-algebras.

4.6.4 Cat-Enriched Lawvere Cat$^{\to}$-Theories

In this section, we try to extend Lawvere $\mathbf{Cat}_o^{\to}$-theories **Fb** through **Fbccc** in Section 4.6 to the **Cat**-enriched Lawvere $\mathbf{Cat}^{\to}$-theories, similarly to Section 4.4.4. The models and morphisms remain the same, but we also have 2-cells. On the one hand, the Lawvere $\mathbf{Cat}_o^{\to}$-Theory for cocomplete fibrations can be **Cat**-enriched. On the other hand, we expect that some fibred structure can not be **Cat**-enriched, for example, complete fibrations and fibred CCCs.

Theorem 4.6.3 (Cloven Fibration)**.** Let **Fb2** be the Lawvere $\mathbf{Cat}^{\to}$-theory freely generated from $(\mathbf{Cat}^{\to})_f^{\mathrm{op}}$ by adding the same new arrows and equations as **Fb**. There exists a bijection between the class of all models for **Fb2** and the class of all models for **Fb**.

Proof. Let $\mathbf{ob}\colon \mathbf{Cat} \to \mathbf{Set}$ be the functor that sends a category to the set of the objects. If (p, T) is a model of **Fb2**, then $(p, \mathbf{ob} \circ T)$ is a model of **Fb**.

Conversely, we prove that for model (p, S) of **Fb**, there exists a unique model (p, T) of **Fb2** such that $\mathbf{ob} \circ T = S$. Here we show that $T\mathbf{F}_1$ is uniquely determined by S. Take an arrow in the category $\mathbf{Cat}^{\to}(\mathbf{cod}, p)$ as follows.

$$\begin{array}{ccccc} \Gamma & \xrightarrow{u} & \Delta & & \psi \\ \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \delta = p\pi} & & \downarrow{\scriptstyle \pi} \\ \Gamma' & \xrightarrow{u'} & \Delta' & & \psi' \end{array}$$

The object part of the functor $T\mathbf{F}_1$ is determined by $S\mathbf{F}_1$. We define an arrow part of the functor $T\mathbf{F}_1$ as follows. Then, $T\mathbf{F}_1$ becomes a functor and (p, T) is a unique model of **Fb2** such that $\mathbf{ob} \circ T = S$.

$$\begin{array}{c} u^*\psi \\ \downarrow{\scriptstyle u'_\gamma[\pi \circ \overline{u}(\psi)]} \\ (u')^*\psi' \end{array}$$

□

Theorem 4.6.4 (Fibred Terminal Object)**.** Let **Fbt2** be the Lawvere $\mathbf{Cat}^{\to}$-theory freely generated from $(\mathbf{Cat}^{\to})_f^{\mathrm{op}}$ by adding the same new arrows and equations as **Fbt**. There exists a bijection between the class of all models for **Fbt2** and the class of all models for **Fbt**.

Proof. The functor for T_1 is uniquely determined so as to send an arrow $\gamma\colon \Gamma \to \Gamma'$ to the following arrow.

$$1_\Gamma \xrightarrow{!^{-1}_{\gamma^* 1_{\Gamma'}}} \gamma^* 1_{\Gamma'} \xrightarrow{\overline{\gamma}(1_{\Gamma'})} 1_{\Gamma'}$$

□

Theorem 4.6.5 (Fibred Finite Limit)**.** Let **Fbfl2** be the Lawvere $\mathbf{Cat}^{\to}$-theory freely generated from $(\mathbf{Cat}^{\to})_f^{\mathrm{op}}$ by adding the same new arrows and equations as **Fbfl**. There exists a bijection between the class of all models for **Fbfl2** and the class of all models for **Fbfl**.

Proof. The functor for P_1 is uniquely determined so as to send arrows

$$\begin{array}{ccc} \psi & \xrightarrow{f} & \varphi \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle r} \\ \psi' & \xrightarrow{f'} & \varphi' \end{array} \qquad \begin{array}{ccc} \sigma & \xrightarrow{g} & \varphi \\ \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle r} \\ \sigma' & \xrightarrow{g'} & \varphi' \end{array} \qquad \begin{array}{c} \Gamma \\ \downarrow{\scriptstyle \gamma = pq = pr = ps} \\ \Gamma' \end{array}$$

to the following arrow (put $\iota = \langle \gamma_{\mathbf{Id}}[q \circ f^* g], \gamma_{\mathbf{Id}}[s \circ g^* f] \rangle_{\gamma^* f', \gamma^* g'}$).

$$\psi \times_{f,g} \sigma \xrightarrow{\iota} (\gamma^* \psi') \times_{\gamma^* f', \gamma^* g'} (\gamma^* \sigma') \xrightarrow{\alpha(\gamma, \psi', \sigma')^{-1}} \gamma^* (\psi' \times_{f', g'} \sigma') \xrightarrow{\overline{\gamma}(\psi' \times_{f', g'} \sigma')} \psi' \times_{f', g'} \sigma'$$

□

Theorem 4.6.6 (Fibred Finite Colimit)**.** Let **Fbfc2** be the Lawvere $\mathbf{Cat}^{\to}$-theory freely generated from $(\mathbf{Cat}^{\to})_f^{\mathrm{op}}$ by adding the same new arrows and equations as **Fbfc**. There exists a bijection between the class of all models for **Fbfc2** and the class of all models for **Fbfc**.

Theorem 4.6.7 (Cocomplete Fibration)**.** Let **Fbco2** be the Lawvere $\mathbf{Cat}^{\to}$-theory freely generated from $(\mathbf{Cat}^{\to})_f^{\mathrm{op}}$ by adding the same new arrows and equations as **Fbco**. There exists a bijection between the class of all models for **Fbco2** and the class of all models for **Fbco**.

Proof. The functor for L_1 is uniquely determined so as to send arrows

$$\begin{array}{ccc} \Gamma & \xrightarrow{u} & \Delta \\ \downarrow{\scriptstyle \gamma = p\pi} & & \downarrow{\scriptstyle \delta} \\ \Gamma' & \xrightarrow{u'} & \Delta' \end{array} \qquad \begin{array}{c} \psi \\ \downarrow{\scriptstyle \pi} \\ \psi' \end{array}$$

to the arrow determined by the following correspondence.

$$
\begin{array}{ccc}
\Sigma_u(\psi) \longrightarrow \Sigma_{u'}(\psi') & \text{over} & \delta \\
\hline
\Sigma_u(\psi) \longrightarrow \delta^*\Sigma_{u'}(\psi') & \text{over} & \mathbf{Id}_\Delta \\
\hline
\psi \longrightarrow u^*\delta^*\Sigma_{u'}(\psi') & \text{over} & \mathbf{Id}_\Gamma \\
\hline
\psi \longrightarrow \gamma^*(u')^*\Sigma_{u'}(\psi') & \text{over} & \mathbf{Id}_\Gamma \\
\hline
\psi \xrightarrow[\pi]{} \psi' \xrightarrow[\eta_{u'}(\psi')]{} (u')^*\Sigma_{u'}(\psi') & \text{over} & \gamma
\end{array}
$$

□

By Definition 3.3.1, we can extend Lawvere $\mathbf{Cat}_o^{\rightarrow}$-theory **GLTS** to the **Cat**-enriched one.

Next, we try to extend **Fbccc** to **Cat**-enriched one. We must define a functor for E_5 that sends arrows

$$
\begin{array}{ccc}
\Gamma & \sigma & \psi \\
\downarrow{\scriptstyle \gamma = ps = p\pi} & \downarrow{\scriptstyle s} & \downarrow{\scriptstyle \pi} \\
\Gamma' & \sigma' & \psi'
\end{array}
$$

to an arrow from $[\sigma, \psi]$ to $[\sigma', \psi']$. Since the algebraic structure for CCC on **Cat** can not be **Cat**-enriched[5], however, we expect there are no such functors. Therefore, we expect there are no **Cat**-enriched Lawvere $\mathbf{Cat}^{\rightarrow}$-theories for which the models are all fibred CCCs. Similarly, we expect there are no **Cat**-enriched Lawvere $\mathbf{Cat}^{\rightarrow}$-theories for which the models are all complete fibrations.

Chapter 5

Conclusion and Future Work

5.1 Conclusion

This thesis gives the notion of abstraction that makes clear the relationship between the soundness and completeness theorems for interpretations of logics and their formula-preservation theorems. Our formulation of interpretations and abstractions is based on algebraic approach. For the formulation, we define a modal fixed point logic $R\mu$. It is considered as a sublogic of modal μ-calculus by way of $L\mu^-$. Free algebra construction on the 2-category $\mathbf{LocOrd}_{lr}$ implies soundness, completeness, and the formula-preservation theorem for $R\mu$. Our formulation includes a new abstraction from a Kripke structure to another structure. Free algebra construction on the 2-category $\mathbf{Cat}^{\rightarrow}$ gives that abstractions of labelled transition systems are determined by ones of labels.

We extend Lawvere theories to Lawvere A-theories and uniformly construct the free algebras of the Lawvere $\mathbf{LocOrd}_{lr}$-theory and the Lawvere $\mathbf{Cat}^{\rightarrow}$-theory.

5.2 Future Work

Finding a suitable algebraic structure for the whole of modal μ-calculus is made difficult due to the fixed-point operators only with the positivity condition. This thesis avoids the problem by restricting the logic and by considering an algebraic structure on the category of locally ordered categories. We expect that the restriction can be lifted by considering a more structured base category other than

LocOrd, for example the category of locally ordered categories with finite products. The algebraic structure should have the least fixed point $\mu(\psi)\colon x \to y$ for arrow $\psi\colon x \times y \to y$. If we can give the algebraic structure and define interpretations, then a completeness result of $L\mu$ for the class of the interpretations should be given by free algebra construction.

Another future work is to analyse examples of **RMu**-algebras and abstractions. Our leading example $\mathbf{Pos_{CL}}$ of **RMu**-algebras consists of simple complete lattices and monotone functions. Instead of $\mathbf{Pos_{CL}}$, we think that complete lattices with some structures and monotone functions which preserve them should form an **RMu**-algebra. In that case, one can naturally define interpretations and abstractions which make good use of such structures.

There may be a relationship between **RMu**-algebras and coalgebras, which have been known as the generalisation of transition systems. Logical relations [22] or lax logical relations [40] may be examples of our abstractions.

It is also future work to refine our theory. In this thesis, we extended some algebraic structures on categories to algebraic structures on 2-categories: the category of categories, the category of locally ordered categories, and the category of functors. Although we extended them separately, the extension may be explained by more general theory.

We may discuss how to give the notion of rule-based presentations itself in general.

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