Semisimplicity, EDPC and discriminator varieties of modal FLew-algebras (Preliminary Version)

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Abstract

In this paper we mention that all semisimple variety of modal $\mathbf{FL}_{ew}$-algebras are discriminator varieties. We also give a characterization of discriminator and EDPC varieties of modal $\mathbf{FL}_{ew}$-algebras follows.

1 Introduction and Preliminaries

In [7], the author proves that all semisimple variety of $\mathbf{FL}_{ew}$-algebras are discriminator varieties.

In this paper, his proof works well also for the variety of modal $\mathbf{FL}_{ew}$-algebras with some modification. We assume a familiarity with the paper [7].

Definition 1. *A modal* $\mathbf{FL}_{ew}$*-algebra is a structure*

(1)$\langle A, \wedge, \vee, 0, 1\rangle$ *is a bounded lattice with the greatest element* 1*, the least element* 0*,*
(2)$\langle A, \cdot, 1\rangle$ *is a commutative monoid,*
(3) $x \cdot y \leq z \Leftrightarrow y \leq x \to z$*, for any* $x, y, z \in A$*, and*
(4) $\Box$ *is a unary operation on* A *satisfying:*

(a) $1 \leq \Box 1$, (b) $\Box x \cdot \Box y \leq \Box(x \cdot y)$, ($c$) $\Box x \leq x$, (d) $\Box x \leq \Box\Box x$, (e) *if* $x \leq y$ *then* $\Box x \leq \Box y$ *for any* $x, y \in A$.

It can be considered the unary operation $\Box$ is S4-like modality. For simplicity, we write xy instead of $x \cdot y$ and abbreviate $\underbrace{x \cdot \ldots \cdot x}_{n}$ by x^n.

The class $\Box\mathcal{FL}_{ew}$ of modal $\mathbf{FL}_{ew}$-algebras is a variety. It is arithmetical, has congruence extension property (CEP) and is congruence 1-regular, i.e., for any congruence θ, the coset of 1 determines θ uniquely. More information about residuated lattices, see the references [10, 6].

In [11], the author shows that the variety of modal $\mathbf{FL}_{ew}$-algebras is generated by its finite simple members. The result is obtained by first shows that every free modal modal $\mathbf{FL}_{ew}$-algebra is semisimple and then showing that every variety generated by a simple modal $\mathbf{FL}_{ew}$-algebra is generated by a set of finite simple modal $\mathbf{FL}_{ew}$-algebras.

To show the former, based on Grishin's idea in [5], and Kowalski and Ono's technique in [9] the author introduced a sequent system $\Box FL_{ew}^{+}$ such that

1. algebras for $\Box FL_{ew}^{+}$ are exactly equal to modal $\mathbf{FL}_{ew}$-algebras,
2. cut elimination theorem holds for $\Box FL_{ew}^{+}$.

Using proof theoretic properties of $\Box FL_{ew}^{+}$, the semisimplicity of free modal $\mathbf{FL}_{ew}$-algebras is obtained. Moreover the author shows that the finite embeddability property of modal $\mathbf{FL}_{ew}$-algebras. Finally, he has the variety of modal $\mathbf{FL}_{ew}$-algebras is generated by its finite simple members. This result can be regarded as a natural generalization of [9].

The following two propositions are folklore and will be used in proofs without further notice. These are terminology in [10], thus we omit the proofs.

Proposition 1. *A modal* $\mathbf{FL}_{ew}$*-algebra is subdirectly irreducible (si) iff it has an element* $a < 1$ *such that for any* $x < 1$ *there exists a positive integer* k *for which* $(\Box x)^k < a$ *holds. It is simple if* a *can be taken to be* 0.

Proposition 2. *In any si modal* $\mathbf{FL}_{ew}$*-algebra,* $\Box x \vee \Box y = 1$ *implies that either* $\Box x = 1$ *or* $\Box y = 1$.

Two sequences of subvarieties of will be of importance in this paper. The first variety is defined relative to , for any $n \in \mathbb{N}$, by the identity:

$$\Box E_n \equiv (\Box x)^n = (\Box x)^{n+1}$$

$$\Box EM_n \equiv \Box x \vee \neg(\Box x)^n = 1$$

A variety has *definable principal congruence* (DPC) iff there is a formula $\phi(x, y, z, t)$ in the first order language of $\mathcal{V}$ with no other free variables than the ones displayed such that, for any $a, b, c, d \in \mathbf{A} \in \mathcal{V}$, we have the following: $(c, d) \in \Theta(a, b)$ iff

$\mathbf{A} \models \phi(a,b,c,d)$. If $\phi(x,y,z,t)$ can be taken a finite set of equations, then $\mathcal{V}$ has *equational definable principal congruence* (EDPC). For more information about DPC and EDPC, see the series of papers [1, 2, 3], and [4].

A variety is semisimple if all its algebras are semisimple, or equivalently, if all its si algebras are simple.

The ternary discriminator is a function t defined by:

$$t(x,y,z) = \begin{cases} x & x = y \\ z & x \neq y \end{cases} \tag{1}$$

Suprisingly, an algebra $\mathbf{A}$ with discriminator term is simple. Indeed, Let $a \neq b$ and $\Theta(a,b)$ a principal congruence. Then $a\Theta(a,b)b$ and for any $c \in \mathbf{A}$, $a = t(a,b,c)\Theta(a,b)t(a,a,c) = c$. Thus, $\Theta(a,b)$ is trivial. Hence $\mathbf{A}$ must be simple.

A discriminator variety is a variety such that the ternary discriminator is a term operation on every si algebra in the variety. All discriminator varieties are arithmetical, has CEP, and semisimple. For a good introduction to discriminator variety, see [12].

The purpose of this paper is to show the conditions correspond precisely to properties of having EDPC and being semisimple variety. The proof of the second of these correspondences involves showing that a variety of modal $\mathbf{FL}_{ew}$-algebras is semisimple iff is a discriminator variety. The first of these correspondences follows from general theorems of EDPC varieties in [1, 2, 3, 4] and is only mentioned here because it provides a ladder for establishing the second one, the proof of which constitutes the bulk of the paper. Since the argument is a little convoluted we ouline the strategy of proof here. A theorem announced in [1] and proved, as Corollary 3.4 in [2] states that if $\mathcal{V}$ is a congruence permutative variety, then $\mathcal{V}$ is discriminator iff $\mathcal{V}$ is semisimple and EDPC. Therefore, in our setting, if we will show that semisimplicity implies EDPC, then we will have proved that semisimplicity is equivalent to being a discriminator variety. Since we know by the first correspondence that satisfying $\Box E_n$ for some $n \in \mathbb{N}$ is equivalent to EDPC, it will suffice to show that semisimplicity implies satisfying $\Box E_n$ for some $n \in \mathbb{N}$. This we will indeed do. The proof is a modification of an argument in [7] which is an argument developed by Kowalski and Kracht to show that all semisimple, finite-type varieties of multi-modal algebras are discriminator varieties [8].

2 Varieties with EDPC

First we will give a characterization of subvarieties $\mathcal{V}$ with EDPC of $\Box\mathcal{FL}_{ew}$.

Theorem 3. *For a variety $\mathcal{V}$ of modal $\mathbf{FL}_{ew}$-algebras, the following conditions are equivalent:*
(i) $\mathcal{V}$ satisfies $(\Box x)^n = (\Box x)^{n+1}$, for some $n \in \mathbb{N}$;
(ii) $\mathcal{V}$ has EDPC;
(iii) $\mathcal{V}$ has DPC;
(iv) $\mathcal{V} \subseteq \mathcal{E}_n$, for some $n \in \mathbb{N}$.

Proof. Let $\Theta(a, b)$ be a principal congruence. Using the correspondence between congruences and filters, we have that $(c, d) \in \Theta(a, b)$ iff $(c \to d)(d \to c)$ belongs to the filter generated by $(a \to b)(b \to a)$ iff $((a \to b)(b \to a))^k \leq (c \to d)(d \to c)$, for some $k \in \mathbb{N}$. Suppose $\mathcal{V}$ satisfies $(\Box x)^n = (\Box x)^{n+1}$, for some $n \in \mathbb{N}$. Then $((a \to b)(b \to a))^{n+1} = ((a \to b)(b \to a))^n$ and hence, $(c, d) \in \Theta(a, b)$ iff $((a \to b)(b \to a))^n \leq (c \to d)(d \to c)$, which can be given as an equation and thus EDPC holds. EDPC trivially yields DPC. Then we have (i) implies (ii) implies (iii). Since (iv) is definitionally equivalent to (i), it remains to show that (iii) implies (i).

Assume the contrary. $\langle \mathbf{A}_n | n \in \mathbb{N}\rangle$ be si algebras from $\mathcal{V}$ such that $\mathbf{A}_n$ falsifies $(\Box x)^n = (\Box x)^{n+1}$ and let $a_n \in \mathbf{A}_n$ be an element which is a witness this fact. Now by DPC, there is a formula $\phi(x, y, z, t)$ first order language of $\mathcal{V}$ such that for all $a, b, c, d \in \mathbf{A} \in \mathcal{V}$ we have $(c, d) \in \Theta(a, b)$ iff $\mathbf{A} \models \phi(a, b, c, d)$. In particular, for any $n \in \mathbb{N}$, $\mathbf{A}_n \models \phi(a_n, 1, a_n^{n+1}, 1)$. This says that a_n^{n+1} belongs to the filter generated by a_n which is always the case. Consider now the ultraproduct $\mathbf{B} = \Pi_{i\in\mathbb{N}}\mathbf{A}_i/U$, for a nonprincipal ultrafilter U on $\mathbb{N}$. Let $a = \langle a_n | n \in \mathbb{N}\rangle/U$, $b = \langle a_n^{n+1} | n \in \mathbb{N}\rangle/U$. Then by the property of ultraproducts $\mathbf{B} \models \phi(a, 1, b, 1)$. By DPC, this means that $(b, 1) \in \Theta(a, 1)$ in other words, that b belongs to the filter generated by a. Therefore $\mathbf{B} \models a^k \leq b$, for some k. However, by the properties of ultraproducts we have that $a^k \leq b$ is false in $\mathbf{B}$ for any k, since for any given k the set $S = \{i \in \mathbb{N} | a_i^k \leq b\} = \{i \in \mathbb{N} | a_i^k \leq a_1^{k+1}\}$ has at most k elements. For on the j-th coordinate. with $j \geq k$, we have $a_j^j \not\leq a_j^{j+1}$ by the choice of a_j and thus $a_j^k \not\leq a_j^{j+1}$ either. Hence S can not belong to any nonprincipal ultrafilter on $\mathbb{N}$. This is a contradiction.

3 Semisimplicity and discriminator

Throughout this section we will assume that $\mathcal{V}$ is a nontrivial semisimple subvariety of $\Box\mathcal{FL}_{ew}$. Consider a modal $\mathbf{FL}_{ew}$-algebra $\mathbf{A}$ in $\mathcal{V}$ and an element $a \in \mathbf{A}\backslash\{1\}$ such that $a^n > 0$ for any $n \in \mathbb{N}$.

3.1 A set up congruence

As an analogue in [7], we introduce a special kind of congruence. Take an algebra $\mathbf{A}$ and an element $a \in A$ as above. Put $\alpha = Cg^{\mathbf{A}}(a, 1)$. By assumption on $\mathbf{A}$, the congruence α is nonzero and nonfull. Since α is principal, there must be a congruence β with β is a subcover of α.

Lemma 4. *There is a positive integer m such that:*
(i) $(\Box a)^{m+1} \equiv_\beta (\Box a)^m$,
(ii) $\neg(\Box a)\ m \equiv_\beta (\neg(\Box a)^m)^2$,
(iii) $(\Box a)^m \equiv_\beta \neg\neg(\Box a)^m$.

Proof. Consider the set $\Delta = \{\theta \in Con(\mathbf{A}) | \theta \geq \beta, \theta \not\geq \alpha\}$. If $\Delta = \{\beta\}$ then $\mathbf{A}/\beta$ is si algebra but not simple, and hence we must have a congruence θ with $\theta \neq \beta$ and $\theta \in \Delta$. By the congruence distributivity, γ defined by $\bigvee \Delta$, is a member of Δ. Therefore $\mathbf{A}/\gamma$ is si algebra hence simple. From this and congruence permutatibity it follows that $\alpha \circ \gamma = 1$. Hence $(1, 0) \in \alpha \circ \gamma$ and there must be an element $c \in A$ with $(1, c) \in \alpha$ and $(c, 0) \in \gamma$. So, $(\neg c, 1) \in \gamma$. $(1, c) \in \alpha$ iff $(\Box a)^m \leq c$ for some $m \in \mathbb{N}$. Thus, $\neg(\Box a)^m \geq \neg c$ and therfore $(\neg(\Box a)^m, 1) \in \gamma$. Then we can assume $c = (\Box a)^m$. By definition we have $\alpha \cap \gamma = \beta$, from which by 1-reguralty we obtain that $1/\alpha \cap 1/\gamma = 1/\beta$.

To prove (i), consider $(\Box a)^{m+1} \vee \neg(\Box a)^m$. By the above argument, we have $\neg(\Box a)^m \in 1/\gamma$ and also $(\Box a)^{m+1} \in 1/\alpha$. It follows that $\neg(\Box a)^m \vee (\Box a)^{m+1} \in 1/\alpha \cap 1/\gamma = 1/\beta$. Thus $\neg(\Box a)^m \vee (\Box a)^{m+1} \equiv_\beta 1$. Therefore $(\neg(\Box a)^m \vee (\Box a)^{m+1})(\Box a)^m \equiv_\beta (\Box a)^m$ as well. Using distributivity over $\cdot, \vee$, we have $(\Box a)^{2m+1} \vee \neg(\Box a)^m(\Box a)^m \equiv_\beta (\Box a)^m$. Hence $(\Box a)^{2m+1} \equiv_\beta (\Box a)^m$. Then $(\Box a)^{m+1} \equiv_\beta (\Box a)^m$.

For (ii) take $(\Box a)^m \vee (\neg(\Box a)^m)^2$ which is in $1/\beta$. Thus $(\Box a)^m \vee (\neg(\Box a)^m)^2 \equiv_\beta 1$. we also get $((\Box a)^m \vee (\neg(\Box a)^m)^2)(\neg(\Box a)^m) \equiv_\beta \neg(\Box a)^m$. After distributing the left-hand side, we obtain $(\neg(\Box a)^m)^3 \equiv_\beta \neg(\Box a)^m$, from which the required $(\neg(\Box a)^m)^2 \equiv_\beta \neg(\Box a)^m$ follows.

Finally, for (iii), we will show first that $(\Box a)^m/\beta$ and $\neg(\Box a)^m/\beta$ are lattice complement in $\mathbf{A}/\beta$, i.e.,

(1) $(\Box a)^m/\beta \vee \neg(\Box a)^m/\beta \equiv_\beta 1$

(2) $(\Box a)^m/\beta \wedge \neg(\Box a)^m/\beta \equiv_\beta 0$.

(1) is clear, since $(\Box a)^m \in 1/\alpha$ and $\neg(\Box a)^m \in 1/\gamma$. For (2) $\neg((\Box a)^m/\beta \vee \neg(\Box a)^m) \equiv_\beta \neg 1 = 0$. From this we have $\neg(\Box a)^m \wedge \neg\neg(\Box a)^m \equiv_\beta 0$. Also we have $\neg\neg(\Box a)^m \geq (\Box a)^m$, thus $\neg(\Box a)^m \wedge (\Box a)^m \equiv_\beta 0$. Next we will prove that if c, d are lattice complements then $\neg c = d$. Since $dc \leq d \wedge c = 0$, using residuation we have $d \leq \neg c$. Also we have $\neg c = (\neg c)1 = (\neg c)(c \vee d) = (\neg c)c \vee (\neg c)d$. Then $\neg c = (\neg c)d \geq \neg c \wedge d$, but we always have $\neg cd \leq \neg c \wedge d$. Then $\neg cd = \neg c \wedge d$. Hence $d \geq \neg c$. Thus we

conclude that $d = \neg c$. Now taking $(\Box a)^m/\beta$ for d and $\neg((\Box a)^m/\beta$ for c which yields the desired $(\Box a)^m \equiv_\beta \neg\neg(\Box a)^m$.

3.2 A property implies semisimplicity

Consider the following condition on $\mathcal{V}$: For every $k \in \mathbb{N}$ there are $r, l \in \mathbb{N}$ such that

$$\mathcal{V} \models (\Box x) \geq (\neg(\neg(\Box x)^r)^k)^l$$

Suppose that $\mathcal{V}$ falsifies the condition above. Then there is a $k \in \mathbb{N}$ such that for all $r, l \in \mathbb{N}$ our variety $\mathcal{V}$ falsifies $(\Box x) \geq (\neg(\neg(\Box x)^r)^k)^l$. Let K be the smallest such k; note that for all $k' \geq K$ the variety $\mathcal{V}$ also falsifies $(\Box x) \geq (\neg(\neg(\Box x)^r)^{k'})^l$.

Let $\mathbf{F}$ be the free algebra in $\mathcal{V}$ on one free generator x. In $\mathbf{F}$, $(\Box x) \not\geq (\neg(\neg(\Box x)^r)^K)^l$ for all $r, l \in \mathbb{N}$. For each $r \in \mathbb{N}$, define θ_r by $Cg^{\mathbf{F}}(\neg(\neg(\Box x)^r)^K, 1)$. Since $\neg(\neg(\Box x)^r)^K \geq \neg(\neg(\Box x)^{r+1})^K$, the family of congruences $\{\theta_r | r \in \mathbb{N}\}$ forms an increasing chain. Put $\alpha = Cg^{\mathbf{F}}(x, 1)$ and $\Theta = \bigvee_{r\in\mathbb{N}} \theta_r$.

Lemma 5. *The congruence Θ lies strictly between 0 and α.*

Proof. If $\Theta = 0$ then consider the one-generated free algebra in $\mathcal{V}$ which is clearly nontrivial algebra. we get that $\mathcal{V} \models \neg(\neg(\Box x)^r)^K = 1$ for all $r \in \mathbb{N}$. In particular, substituting 0 for x. we obtain that $\mathcal{V} \models \neg(\neg(\Box 0)^r)^K = 1$. This means that $\mathcal{V} \models 0 = 1$, contradicting the nontriviality assumption.

To see that $\Theta \leq \alpha$ notice that $(x, 1) \in \alpha$ by definition, therefore for any $r \in \mathbb{N}$ we have $(\Box x)^r \equiv_\alpha 1$ and also $(\neg(\neg(\Box x)^r)^K, 1) \in \alpha$. This means that the pair generating θ_r belongs to α and $\theta_r \leq \alpha$ for all $r \in \mathbb{N}$. Hence $\Theta = \bigvee_{r\in\mathbb{N}} \theta_r \leq \alpha$. Since α is principal, iff there is a finite set $S \subseteq \{\theta_r | r \in \mathbb{N}\}$ with $\bigvee S = \alpha$. However, as $\{\theta_r | r \in \mathbb{N}\}$ is an increasing chain. If such S exists then there is an $r \in \mathbb{N}$ such that $\theta_r = \alpha$. Both θ_r and α is principal and thus $\theta_r = \alpha$ iff there exists an $l \in \mathbb{N}$ with $\Box x \geq (\neg(\neg(\Box x)^r)^K)^l$. This contradicts the assumption that $\mathcal{V} \not\models (\Box x) \geq (\neg(\neg(\Box x)^r)^k)^l$

Lemma 6. *$\mathcal{V}$ satisfies $(\Box x) \geq (\neg(\neg(\Box x)^r)^k)^l$.*

Proof. Suppose the contrary, we assume that $\mathcal{V} \not\models (\Box x) \geq (\neg(\neg(\Box x)^r)^k)^l$. Let K be the smallest witness which holds the previous lemma and β a congruence with $\Theta \leq \beta < \alpha$. By Lemma 4, we have $\neg(\Box a)^m \equiv_\beta (\neg(\Box a)^m)^2$ for some $m \in \mathbb{N}$. Thus $\neg(\Box a)\ m \equiv_\beta (\neg(\Box a)^m)^K$ as well. Therefore using Lemma 4 again, we obtain that $(\Box a)^m \equiv_\beta \neg\neg(\Box a)^m \equiv_\beta \neg(\neg(\Box a)^m)^K \equiv_{\theta_m} 1$. Since $\theta_m \leq \Theta \leq \beta$, this yields $(\Box a)^m \equiv_\beta 1$. Therefore also $\Box a \equiv_\beta 1$ and thus $\beta \geq \alpha$ which contradicts the choice of β.

3.3 An ultraproduct construction

We assume that our subvariety $\mathcal{V}$ of $\Box\mathcal{FL}_{ew}$ satisfies $(\Box x) \geq (\neg(\neg(\Box x)^r)^k)^l$. Define a function $r : \mathbb{N} \to \mathbb{N}$ by taking $r(i)$ to be the smallest number such that there exists an $l \in \mathbb{N}$ with $\mathcal{V} \models (\Box x) \geq (\neg(\neg(\Box x)^{r(i)})^i)^l$.

Lemma 7. *the function r defined above is nondecreasing.*

Proof. Suppose the contrary. Then for a certain $i \in \mathbb{N}$, we have $r(i) > r(i+1)$. We also have $(\neg(\Box x)^{r(i+1)})^{i+1}) \leq (\neg(\Box x)^{r(i+1)})^i)$ and thus $\neg(\neg(\Box x)^{r(i+1)})^{i+1}) \geq \neg(\neg(\Box x)^{r(i+1)})^i)$. Hence by definition of the function r there is an $l \in \mathbb{N}$ such that $\Box x \geq (\neg(\neg(\Box x)^{r(i+1)})^{i+1})^l$ holds. Taking the l-th power of both sides of the previous inequality, we obtain $(\neg(\neg(\Box x)^{r(i+1)})^{i+1}))^l \geq (\neg(\neg(\Box x)^{r(i+1)})^i))^l$, and this in turn yields $\Box x \geq (\neg(\neg(\Box x)^{r(i+1)})^i)^l$. However it is a contradiction, since $r(i)$ is the smallest number for which a suitable l exists, yet $r(i+1)$ is strictly smaller.

Next, we define another function $l : \mathbb{N} \to \mathbb{N}$ by taking $l(i)$ to be the smallest number such that $\mathcal{V} \models (\Box x) \geq (\neg(\neg(\Box x)^{r(i)})^i)^{l(i)}$. Thus, defined l depends on i via r.

Lemma 8. *If $\mathcal{V}$ falsifies $(\Box x)^{n+1} = (\Box x)^n$ for all $n \in \mathbb{N}$ then for each $i > 0$ there is a simple modal $\mathbf{FL}_{ew}$-algebra $\mathbf{A}_i$ and an element $a_i \in A_i$ such that $(\Box a_i)^{r(i)} > 0$ but $(\Box a_i)^{2r(i)} = 0$.*

Proof. Suppose otherwise. Then there exists a $j > 0$ for each simple algebra $\mathbf{A} \in \mathcal{V}$ and each element $a \in A$ we have that $(\Box a)^{2r(i)} = 0$ implies $(\Box a)^{r(i)} = 0$. Take any $b < 1$. As $\mathbf{A}$ is simple we must have a $k \in \mathbb{N}$ with $(\Box b)^k = 0 < (\Box b)^{k-1}$. We will show that $(\Box b)^{r(j)} = 0$. If $k \leq 2r(j)$ then this is clear. So suppose $k > 2r(j)$ then let s be the smallest integer with $2s \cdot r(j) \geq k > k - 1s \cdot r(j)$. Hence $(\Box b)^{2s \cdot r(j)} \leq (\Box b)^k = 0 \leq (\Box b)^{k-1} \leq (\Box b)^{s \cdot r(j)}$ and it follows that $((\Box b)^s)^{2r(j)} = 0 < ((\Box b)^s)^{r(j)}$. However the implication above is universally quantified. $((\Box b)^s)^{2r(j)} = 0$ forces $((\Box b)^s)^{r(j)} = 0$ which is a contradiction. Thus we obtain that all simple algebras in $\mathcal{V}$ satisfy $((\Box b)^s)^{r(j)+1} = ((\Box b)^s)^{r(j)} = 0$ for all nonunit element. Since $1^{r(j)+1} = 1^{r(j)}$, it follows that $\mathcal{V}$ satisfies $(\Box x)^{r(j)+1} = (\Box x)^{r(j)}$ contrary to the assumption.

The following we put $b_i = \neg(\Box a_i)^{r(i)}$ and $f_i = l(i) \cdot 4r(i)$.

Lemma 9. *For every positive integer k, the following holds: $b_i^k > 0$ and $(\neg b_i^k)^{f(k)} = 0$, for every $i \geq k$.*

Proof. To see the first statement, suppose that $b_i^k = 0$. Thus $b_i^i = 0$ as $k \leq i$. By definition of b_i then, we have $b_i^i = (\neg(\Box a_i)^{r(i)})^i = 0$ and thus $\neg(\neg(\Box a_i)^{r(i)})^i = 1$.

Hence $(\neg(\neg(\Box a_i)^{r(i)})^i)^l = 1$ for any $l \in \mathbb{N}$, therefore $\Box a_i \not\geq (\neg(\neg(\Box a_i)^{r(i)})^i)^l$ for any $l \in \mathbb{N}$. This contradicts the definition of r.

For the second statement, we can reason as follows. We have $r(i) \geq |r(i)/r(k)| \cdot r(k)$, therefore

$$(\Box a_i)^{r(i)} \leq (\Box a_i)^{|r(i)/r(k)|\cdot r(k)},$$

$$(\neg(\Box a_i)^{r(i)})^k \geq (\neg(\Box a_i)^{|r(i)/r(k)|\cdot r(k)}),$$

$$(\neg(\neg(\Box a_i)^{r(i)})^k)^{f(k)} \leq (\neg(\neg(\Box a_i)^{|r(i)/r(k)|\cdot r(k)}))^{f(k)}.$$

On the other hand, from $(\neg(\neg(\Box x)^{r(k)})^k)^{l(k)} \leq x$ we can have

$$(\neg(\neg(\Box a_i)^{|r(i)/r(k)|\cdot r(k)})^k)^{f(k)} \leq (\Box a_i)^{|r(i)/r(k)|\cdot 4r(k)}.$$

By Lemma 7, we have $|r(i)/r(k)| \geq 1$ and hence, $|r(i)/r(k)| \cdot 4r(k) \geq 2r(i)$. We obtain that $(\Box a_i)^{|r(i)/r(k)|\cdot 4r(k)} \leq (\Box a_i)^{2r(i)} = 0$ and then putting all the inequalities together we finally get $(\neg b_i^k)^{f(k)} = (\neg(\neg(\Box a_i)^{r(i)})^k)^{f(k)} \leq (\Box a_i)^{|r(i)/r(k)|\cdot 4r(k)} \leq (\Box a_i)^{2r(i)} = 0$, as required by the claim.

Next let $\mathbf{B}$ be the ultraproduct $\Pi_{i\in\mathbb{N}}\mathbf{A}_i/U$, for a nonprincipal ultrafilter U on $\mathbb{N}$. Consider the element $b = \langle b_i | i \in \mathbb{N}\rangle/U$. The next lemma follows from previous lemma by properties of ultraproducts.

Lemma 10. *For any $k \in \mathbb{N}$, $\mathbf{B} \models b^k > 0$ and $\mathbf{B} \models (\neg b_i^k)^{f(k)} = 0$.*

3.4 The main result

Lemma 11. *The variety $\mathcal{V}$ satisfies $(\Box x)^{n+1} = (\Box x)^n$ for some $n \in \mathbb{N}$.*

Proof. Suppose $\mathcal{V}$ falsifies $(\Box x)^{n+1} = (\Box x)^n$ for all $n \in \mathbb{N}$. Suppose that $\mathbf{B} \in \mathcal{V}$ and $b \in \mathbf{B}$ such that $b^k > 0$ and $(\neg b^k)^{f(b)} = 0$ for all $k \in \mathbb{N}$. Then putting $\alpha = Cg^{\mathbf{B}}(b, 1)$ and taking β and γ as in our setup congruences, By Lemma 4, we have that there is an $m \in \mathbb{N}$ for which $\neg b^m \equiv_\beta (\neg b^m)^2$. Therefore $\neg b^m \equiv_\beta (\neg b^m)^{f(m)} = 0$ and thus $b^m \equiv_\beta$. it follows that $b \equiv_\beta 1$, which forces $\beta \geq \alpha$. This contradicts the choice of β.

Theorem 12 (Main theorem)**.** *For a variety $\mathcal{V}$ of modal $\mathbf{FL}_{ew}$-algebras the following conditions are equivalent:*
(i) $\mathcal{V}$ satisfies $\Box x \vee \neg(\Box x)^n$, for some $n \in \mathbb{N}$;
(ii) $\mathcal{V}$ is semisimple;
(iii) $\mathcal{V}$ is a discriminator variety.

Proof. (iii) implies (ii) is well-known. To see that (i) implies (ii), let $\mathbf{A}$ be a si algebra in $\mathcal{V}$. For any element $a \in \mathbf{A}$, either $a = 1$ or $(\Box a)^n = 0$ for some $n \in \mathbb{N}$. Then it is easy to see that the term $t(x, y, z) = ((\Box(x \leftrightarrow y)^n \wedge z)) \vee (\neg(\Box(x \leftrightarrow y)^n) \wedge x)$ is a discriminator term for $\mathcal{V}$.

It remains to show that (ii) implies (i). Take any simple algebra $\mathbf{A}$ in $\mathcal{V}$ and an element $a \in A$ with $a < 1$. By the previous lemma, there exists an $n \in \mathbb{N}$ such that $(\Box a)^{n+1} = (\Box a)^n$. If $(\Box a)^n > 0$ then $\{x \in A | x \geq (\Box a)^n\}$ is a filter which is closed under multiplication, defines a nontrivial and nonfull congruence on $\mathbf{A}$ and a has been chosen arbitrarily, we obtain that $\mathcal{V}$ verifies $\Box x \vee \neg(\Box x)^n$ as desired.

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