A note on the weak topology for the constructive completion of the space D(R)

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A note on the weak topology for the constructive completion of the space $\mathcal{D}(\mathbb{R})$

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1 Introduction

The space $\mathcal{D}(\mathbb{R})$ consists of test functions (infinitely differentiable functions on $\mathbb{R}$ with compact support), and has the locally convex structure defined by the seminorms

$$p_{\alpha,\beta}(\phi) := \sup_n \max_{l \leq \beta(n)} \sup_{|x| \geq n} 2^{\alpha(n)} |\phi^{(l)}(x)| \quad (\phi \in \mathcal{D}(\mathbb{R}), \alpha, \beta \in \mathbb{N} \to \mathbb{N}).$$

This space is an important example of a non-metrizable locally convex space. A distribution (or generalized function) is also a sequentially continuous linear functional on $\mathcal{D}(\mathbb{R})$.

The space $\mathcal{D}(\mathbb{R})$ is complete in classical mathematics (CM), and its dual space $\mathcal{D}^*(\mathbb{R})$ (the space of distributions) is also weakly complete. On the other hand, we have some results for the completeness properties in constructive mathematics as follows.

- The completeness of the space $\mathcal{D}(\mathbb{R})$ is equivalent to the principle BD-$\mathbb{N}$, which can be proved in Brouwer's intuitionistic mathematics (INT) and constructive recursive mathematics of Markov's school (CRM), but not in Bishop's constructive mathematics (BISH)[3, Theorem 4].
- The weak completeness of the dual space $\mathcal{D}^*(\mathbb{R})$ can be proved in the three frameworks [5, Theorem 4.10].

Thus, in BISH, the dual space $\mathcal{D}^*(\mathbb{R})$ is weakly complete although $\mathcal{D}(\mathbb{R})$ is not complete. These matters also mean that the completeness of $\mathcal{D}(\mathbb{R})$ is not necessary for proving the weak completeness in the three frameworks and classical mathematics, although it is required in many classical proofs.

Here BISH, INT and CRM are known as main frameworks of constructive mathematics (see [4, Ch.1] for more details). BISH is mathematics with intuitionistic logic, and all theorems in BISH are ones in CM. On the other hand, there exist some theorems in CM but not in BISH (e.g. Principle of Excluded Middle). INT and CRM can be obtained by adding some axioms to BISH, respectively (see [4, Ch.4] for more details). All theorems in BISH are therefore ones in INT and CRM. Theses two frameworks have some theorems whose negation can be proved in CM (see Fig.1), and the converse holds.

Now the weak completeness of $\mathcal{D}^*(\mathbb{R})$ can be proved in BISH, by showing the following matters (see [5] for more details).

1. The completion $\widetilde{\mathcal{D}}(\mathbb{R})$ of $\mathcal{D}(\mathbb{R})$.
2. The weak completeness of the dual space $\widetilde{\mathcal{D}}^*(\mathbb{R})$ (the space of sequentially continuous linear functionals on $\widetilde{\mathcal{D}}(\mathbb{R})$)
3. Extension of a distribution to $\widetilde{\mathcal{D}}(\mathbb{R})$.

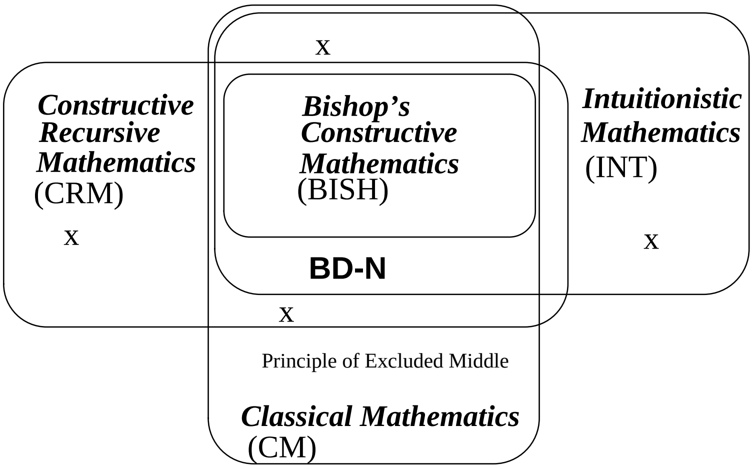

Fig.1 The sets of theorems in the four frameworks

(The letter "x" denotes a theorem which can be proved in one framework but cannot in another one. Principle of Excluded Middle is a theorem only in CM. BD-$\mathbb{N}$ is in particular in CM, INT and CRM, but not in BISH.)

Note that, in BISH, $\widetilde{\mathcal{D}}(\mathbb{R})$ is equal to $\mathcal{D}(\mathbb{R})$ as sets if and only if BD-$\mathbb{N}$ can be proved, and therefore $\widetilde{\mathcal{D}}(\mathbb{R})$ is not equal to $\mathcal{D}(\mathbb{R})$. For (3), a distribution can be uniquely extended to $\widetilde{\mathcal{D}}(\mathbb{R})$ [5, Theorem 4.9]. We therefore see that $\widetilde{\mathcal{D}}^*(\mathbb{R})$ is equal to $\mathcal{D}^*(\mathbb{R})$ as sets in BISH, although $\widetilde{\mathcal{D}}(\mathbb{R})$ is not equal to $\mathcal{D}(\mathbb{R})$. Moreover, is $\widetilde{\mathcal{D}}^*(\mathbb{R})$ topologically equivalent to $\mathcal{D}^*(\mathbb{R})$? In this paper, we show that $\widetilde{\mathcal{D}}^*(\mathbb{R})$ is equivalent to $\mathcal{D}^*(\mathbb{R})$ with respect to convergence in BISH.

2 Preliminary

Let X be a vector space over $\mathbb{R}$. A mapping $p : X \to \mathbb{R}^{0+}$ is a *seminorm* on X if it satisfies that for $x, y \in X$ and $\lambda \in \mathbb{R}$, (1) $p(x+y) \leq p(x) + p(y)$ and (2) $p(\lambda x) = |\lambda| p(x)$. A pair $(X, \{p_i\})$ is *locally convex space* over $\mathbb{R}$ if for all index i and x in X, whenever $p_i(x) = 0$ then $x = 0$. $\{x_n\}$ *converges* to x in X if $\forall k \in \mathbb{N} \forall i \in I \exists N \in \mathbb{N} \left[n \geq N \implies p_i(x - x_n) < 2^{-k} \right]$. Let u be a linear functional on X. u is *sequentially continuous* on X if for each sequence $\{x_n\}$ in X and $x \in X$ $\{x_n\}$ converges to x in X the sequence $\{u(x_n)\}$ converges to $u(x)$ in $\mathbb{R}$. The *dual space* X^* *with weak topology* of a locally convex space $(X, \{p_i\})$ is a locally convex space of sequentially continuous linear functionals on X, with the seminorms $\{\| \cdot \|_x\}$ defined by $\|u\|_x := |u(x)| \quad (x \in X)$.

We say that a subset A of $\mathbb{N}$ is *pseudobounded* if for any sequence $\{a_n\}$ in A, $a_n < n$ for all sufficiently large n. Any bounded subset of $\mathbb{N}$ is pseudobounded. On the other hand, the following principle cannot be proved in BISH [1]:

BD-$\mathbb{N}$ Every countable pseudobounded subset of $\mathbb{N}$ is bounded.

Thus the converse cannot be proved in BISH. We can however prove BD-$\mathbb{N}$ in CM, INT and CRM [2].

For a function $f : \mathbb{R} \to \mathbb{R}$, let supp f be the closure of the set $\{x \in \mathbb{R} : |f(x)| > 0\}$ in $\mathbb{R}$, and set

$$\mathrm{supp}_{\mathbb{N}} f \quad := \quad \{0\} \cup \{n \in \mathbb{N} : \exists q \in \mathbb{Q}(|q| \geq n \wedge |f(q)| > 0\}.$$

A function f is said to *have compact support* if the set supp f is bounded. It is easy to show that a sequentially continuous function f has compact support if and only if the set $\mathrm{supp}_{\mathbb{N}} f$ is bounded. We say that f *has pseudobounded support*, if the set $\mathrm{supp}_{\mathbb{N}} f$ is pseudobounded. A *test function* is an infinitely differentiable function on $\mathbb{R}$. The space $\widetilde{\mathcal{D}}(\mathbb{R})$ is the locally convex space of infinitely differentiable functions with pseudobounded support, taken with the seminorms $\{p_{\alpha,\beta}\}$ of $\mathcal{D}(\mathbb{R})$. Note that every $p_{\alpha,\beta}$ can be defined in $\widetilde{\mathcal{D}}(\mathbb{R})$ [5, Proposition 2.7].

Let u be a linear functional on $\mathcal{D}(\mathbb{R})$. We often write $\langle u, \phi\rangle := u(\phi)$ for any ϕ in $\mathcal{D}(\mathbb{R})$ (or $\widetilde{\mathcal{D}}(\mathbb{R})$). Set

$$\begin{aligned} \text{supp } u &:= \left\{x \in \mathbb{R} : \forall k \exists \phi \in \mathcal{D}(\mathbb{R})[\text{supp } \phi \subset (x - 2^{-k}, x + 2^{-k}) \wedge |\langle u, \phi\rangle| > 0]\right\}, \\ \text{supp}_{\mathbb{N}} u &:= \{0\} \cup \{n \in \mathbb{N} : \exists x \in \mathbb{R}[|x| > n \wedge x \in \text{supp } u]\}. \end{aligned}$$

We can then define compact support and pseudobounded one of a distribution, similarly to a function $\mathbb{R}$ into $\mathbb{R}$. Note that for each linear functional u on $\mathcal{D}(\mathbb{R})$ and ϕ in $\mathcal{D}(\mathbb{R})$, if $|\langle u, \phi\rangle| > 0$, then we have supp $u \cap$ supp $\phi \neq \emptyset$[1] [6, Proposition 5.1].

3 The dual spaces $\mathcal{D}^*(\mathbb{R})$ and $\widetilde{\mathcal{D}}^*(\mathbb{R})$

The spaces $\widetilde{\mathcal{D}}^*(\mathbb{R})$ and $\mathcal{D}^*(\mathbb{R})$ are equal as sets, from the following result:

Theorem 1. *[5, Theorem 4.9] Every distribution is uniquely extended to* $\widetilde{\mathcal{D}}(\mathbb{R})$.

We moreover show that these spaces are equivalent with respect to convergence.

Theorem 2. *A sequence* $\{u_n\}$ *of distributions converges to* 0 *in* $\mathcal{D}^*(\mathbb{R})$ *if and only if it does in* $\widetilde{\mathcal{D}}^*(\mathbb{R})$.

Proof. The part "if" is trivial. We show the part "only if". Assume that a sequence $\{u_n\}$ converges to 0 in $\mathcal{D}^*(\mathbb{R})$. Let k be any natural number, and ϕ any element in $\widetilde{\mathcal{D}}(\mathbb{R})$. Choose a sequence $\{\rho_n\}$ of test functions such that

- for each n, $0 \leq \rho_n(x) \leq 1$ if $x \in \mathbb{R}$,
- for each n, $\sup_{n-1 \leq |x| \leq n+1} \rho_n(x) > 0$,
- for each n, $\rho_n(x) = 0$ if $|x| \leq n - 1$ or $n + 1 \leq |x|$, and
- $\sum_{i=0}^{\infty} \rho_n(x) = 1$ for all $x \in \mathbb{R}$,

given in [5, Lemma 4.4]. Construct a binary sequence $\{\lambda_n\}$ such that $\lambda_0 = 0$ and

$$\lambda_n = 0 \implies \left|\sum_{i=n+1}^{\infty} \langle u_n, \rho_i\phi\rangle\right| < 2^{-(k+1)}, \quad \lambda_n = 1 \implies \left|\sum_{i=n+1}^{\infty} \langle u_n, \rho_i\phi\rangle\right| > 0.$$

Define a sequence $\{a_n\}$ in $\text{supp}_{\mathbb{N}}\phi$ as follows: if $\lambda_n = 0$, then set $a_n := 0$; if $\lambda_n = 1$, then there exists m in $\text{supp}_{\mathbb{N}}\rho_{m+1}\phi$ such that $m \geq n$ and supp $u_n \cap$ supp $\rho_{m+1}\phi \neq \emptyset$ from [6, Proposition 5.1], and then set $a_n := m$. Since the set $\text{supp}_{\mathbb{N}}\phi$ is pseudobounded and since the set $\text{supp}_{\mathbb{N}}\rho_i\phi$ is a subset of $\text{supp}_{\mathbb{N}}\phi$ for all i, there exists N such that $a_n < n$ for all $n \geq N$. If $\lambda_n = 1$ for some $n \geq N$, then there exists m in $\mathbb{N}$ such that $n \leq m = a_n < N \leq n$, a contradiction. Thus we have $\left|\sum_{i=n+1}^{\infty} \langle u_n, \rho_i\phi\rangle\right| < 2^{-(k+1)}$ for all $n \geq N$.

We can here take M in $\mathbb{N}$ such that $M \geq N$ and $\left|\sum_{i=0}^{N} \langle u_n, \rho_i\phi\rangle\right| = \left|\left\langle u_n, \sum_{i=0}^{N} \rho_i\phi\right\rangle\right| < 2^{-(k+1)}$ for all $n \geq M$, since $\{u_n\}$ converges in $\mathcal{D}^*(\mathbb{R})$. Then, for all $n \geq M$, we have

$$|\langle u_n, \phi\rangle| = \left|\left\langle u_n, \sum_{i=0}^{\infty} \rho_i\phi\right\rangle\right| \leq \left|\sum_{i=0}^{N} \langle u_n, \rho_i\phi\rangle\right| + \left|\sum_{i=N+1}^{\infty} \langle u_n, \rho_i\phi\rangle\right| < 2^{-(k+1)} + 2^{-(k+1)} = 2^{-k}.$$

□

We say that a subset S of a locally convex space X is *closed*, if for each x in X if there exists a sequence $\{x_n\}$ in S conversing to x, then $x \in S$. We immediately obtain the following proposition.

[1]For a set A, we write $A \neq \emptyset$ if we can construct an element of A.

Proposition 3. *A set S of distributions is closed in $\mathcal{D}^*(\mathbb{R})$ if and only if it is in $\widetilde{\mathcal{D}}^*(\mathbb{R})$.*

Thus $\mathcal{D}^*(\mathbb{R})$ and $\widetilde{\mathcal{D}}^*(\mathbb{R})$ are equivalent with respect to closed sets, in that sense.

Can we moreover show the equivalence with respect to fundamental neighbourhoods of 0? That is, does the following hold? For all ϕ in $\widetilde{\mathcal{D}}(\mathbb{R})$ and k in $\mathbb{N}$, there exist ψ in $\mathcal{D}(\mathbb{R})$ and m in $\mathbb{N}$ such that for all u in $\mathcal{D}^*(\mathbb{R})$,

$$|\langle u, \psi \rangle| < 2^{-m} \implies |\langle u, \phi \rangle| < 2^{-k}.$$

In the proof of Theorem 2, we construct a required ψ in $\mathcal{D}(\mathbb{R})$ from a given ϕ in $\widetilde{\mathcal{D}}(\mathbb{R})$, by investigating every u_n. But we cannot solve this problem by a straightforward modification of this way, since the union $\cup V_{\phi,k} \equiv \left\{u \in \mathcal{D}^*(\mathbb{R}) : \exists k \in \mathbb{N} \exists \phi \in \mathcal{D}(\mathbb{R}) \left[\|u\|_\phi < 2^{-k}\right]\right\}$ of all fundamental neighbourhoods of 0 is not countable. Another way may be necessary for this problem.

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