Algebraic Structures for Cocomplete Fibrations and Fibred CCCs

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Abstract. We give an algebraic structure for which the models are all cloven fibrations. We take $\mathbf{Cat}^{\rightarrow}$ as the base category. The algebraic structure is presented by using the generalised Lawvere theories which Nishizawa and Power introduced. Next, we extend it to make the models cloven fibrations with some fibred structures. There exist **Set**-enriched algebraic structures for cloven fibrations with fibred finite colimits, cloven fibrations with fibred finite limits, cocomplete fibrations, complete fibrations, or fibred CCCs. On the one hand, there exist **Cat**-enriched algebraic structures for cloven fibrations with fibred finite colimits, cloven fibrations with fibred finite limits, or cocomplete fibrations. On the other hand, we expect there exist no **Cat**-enriched ones for complete fibrations or fibred CCCs.

1 Introduction

The correspondence between finitary monads on **Set** and finitary algebraic theories (i.e., collections of basic operations and equations such that the arity of each basic operation is finite) is one of the deepest relationship in category theory [7]. Given a finitary algebraic theory, a term model of the theory is a free algebra of the corresponding finitary monad. The correspondence is a framework to give semantics of various logics or programming languages.

The notion of finitary algebraic theory is formalised as Lawvere theory [1]. The Lawvere theory corresponding to a finitary monad is invariant, up to the simple and obvious notion of isomorphism between them. In the paper [6], Nishizawa and Power generalise the correspondence between Lawvere theories and finitary

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monads on **Set** in two ways. First, we allow our theories to be enriched in a category V that is locally finitely presentable as a symmetric monoidal closed category. And second, we allow the arities of our theories to be finitely presentable objects of a locally finitely presentable V-category A. We call the resulting notion a Lawvere A-theory. Lawvere theories are instances of Lawvere A-theory where $V = A = \mathbf{Set}$.

When $V = \mathbf{Set}$ and $A = \mathbf{Cat}_o$, there exist Lawvere $\mathbf{Cat}_o$-theories for which the models are all categories with terminal objects, finite limits, or exponents, respectively. When $V = A = \mathbf{Cat}$, there exist Lawvere **Cat**-theories for which the models are all categories with terminal objects or finite limits, respectively. Unfortunately, however, the notion of fibration seems not to be monadic over **Cat** or any integer power of **Cat**, as discussed in the paper [7].

In this paper, we solve this problem by putting $A = \mathbf{Cat}^{\rightarrow}$ or $A = \mathbf{Cat}_o^{\rightarrow}$. Putting $V = \mathbf{Set}$ and $A = \mathbf{Cat}_o^{\rightarrow}$, we give Lawvere $\mathbf{Cat}_o^{\rightarrow}$-theories for which the models are all cloven fibrations with fibred terminal objects, all cloven fibrations with fibred finite limits, all cloven fibrations with fibred finite colimits, all cocomplete fibrations, all complete fibrations, and all fibred CCCs, respectively. Putting $V = \mathbf{Cat}$ and $A = \mathbf{Cat}^{\rightarrow}$, we give Lawvere $\mathbf{Cat}^{\rightarrow}$-theories for which the models are all cloven fibrations with fibred terminal objects, all cloven fibrations with fibred finite limits, all cloven fibrations with fibred finite colimits, and all cocomplete fibrations, respectively.

This paper is organised as follows. In Section 2, we define six properties for a functor and prove that a functor satisfies them if and only if it is a cloven fibration. In Section 3, we define a Lawvere $\mathbf{Cat}_o^{\rightarrow}$-theory for which the models are all cloven fibrations. In Section 4, we extend it to make the models cloven fibrations with some fibred structures. In Section 5, we extend some of them to **Cat**-enriched ones.

2 Cloven Fibrations

In this section, we define six properties and prove that a functor $p\colon E \to B$ satisfies them if and only if p is a cloven fibration.

First, we show definitions [4, 3] for cloven fibration. Let $\mathbf{E}$ and $\mathbf{B}$ be arbitrary categories and let $p\colon \mathbf{E} \to \mathbf{B}$ be a functor. We write **Id** for identity arrows.

Definition 1. *An arrow $c\colon \tau \to \psi$ in $\mathbf{E}$ is a* cartesian *over $u = pc\colon \Gamma \to \Delta$ in $\mathbf{B}$ if, for every $f\colon \varphi \to \psi$ with $pf = u \circ v$ in $\mathbf{B}$, there exists a unique arrow $g\colon \varphi \to \tau$ in $\mathbf{E}$ such that $pg = v$ and $f = c \circ g$.*

Definition 2. *A functor $p\colon \mathbf{E} \to \mathbf{B}$ is a* fibration *if, for all ψ in $\mathbf{E}$ and arrows $u\colon \Gamma \to p\psi$ in $\mathbf{B}$, there exists an object τ in $\mathbf{E}$ and a cartesian $c\colon \tau \to \psi$ over u.*

Definition 3. *If $p\colon \mathbf{E} \to \mathbf{B}$ is a fibration, a* cleavage *for p is a particular choice of cartesian $\overline{u}(\psi)\colon u^*\psi \to \psi$ for each ψ, u. A fibration equipped with a* particular *cleavage is called a* cloven fibration.

Definition 4. *For each object* Γ *in* $\mathbf{B}$*, the* fibre $\mathbf{E}_\Gamma$ *over* Γ *is the following category.*

- *object: object* ψ *in* $\mathbf{E}$ *such that* $p\psi = \Gamma$
- *arrow: arrow* f *in* $\mathbf{E}$ *such that* $pf = \mathbf{Id}_\Gamma$

Definition 5. *If* $p\colon \mathbf{E} \to \mathbf{B}$ *is a cloven fibration,* $u\colon \Gamma \to \Delta$ *in* $\mathbf{B}$ *determines a* reindexing functor $u^*\colon \mathbf{E}_\Delta \to \mathbf{E}_\Gamma$ *as follows.*

- *on objects, for* ψ *in* $\mathbf{E}_\Delta$*,* $u^*\psi$ *is the domain of the cartesian for* ψ *and* u *given by the cleavage.*
- *on arrows, for* $f\colon \varphi \to \psi$ *in* $\mathbf{E}_\Delta$*,* u^*f *is determined as the unique arrow that* $p(u^*f) = \mathbf{Id}$ *and* $\overline{u}(\psi) \circ u^*f = f \circ \overline{u}(\varphi)$*.*

Next, we define six properties F_1 through F_6. We also show rule-based presentation of each property. A judgement $\Gamma\colon$ **base** represents that Γ is an object of $\mathbf{B}$. A judgement $u\colon \Gamma \to \Delta$ represents that u is an arrow from Γ to Δ in $\mathbf{B}$. A judgement $\Gamma \vdash \psi$ represents that ψ is an object of $\mathbf{E}$ such that $p\psi = \Gamma$. A judgement $u\colon \Gamma \to \Delta \vdash f\colon \psi \to \varphi$ represents that $f\colon \psi \to \varphi$ is an arrow of $\mathbf{E}$ such that $p\psi = \Gamma$, $p\varphi = \Delta$, and $pf = u$. A judgement $u\colon \Gamma \to \Delta \vdash f = g\colon \psi \to \varphi$ represents that $f, g\colon \psi \to \varphi$ are arrows of $\mathbf{E}$ such that $p\psi = \Gamma$, $p\varphi = \Delta$, $pf = u$, $pg = u$, and $f = g$. We write $\Gamma \vdash f\colon \psi \to \varphi$ for $\mathbf{Id}\colon \Gamma \to \Gamma \vdash f\colon \psi \to \varphi$.

A functor p satisfies F_1 if for arrow $u\colon \Gamma \to \Delta$ in B and object ψ in E such that $p\psi = \Delta$, there exists an object $u^*\psi$ in E such that $p(u^*\psi) = \Gamma$.

$$\frac{u\colon \Gamma \to \Delta \quad \Delta \vdash \psi}{\Gamma \vdash u^*\psi}\ F_1$$

A functor p satisfies F_2 if for arrow $u\colon \Gamma \to \Delta$ in B and object ψ in E such that $p\psi = \Delta$, there exists an arrow $\overline{u}(\psi)\colon u^*\psi \to \psi$ in E such that $p(\overline{u}(\psi)) = u$.

$$\frac{u\colon \Gamma \to \Delta \quad \Delta \vdash \psi}{u\colon \Gamma \to \Delta \vdash \overline{u}(\psi)\colon u^*\psi \to \psi}\ F_2$$

A functor p satisfies F_3 if for arrow $v\colon \Lambda \to \Gamma$, $u\colon \Gamma \to \Delta$ in B and arrow $f\colon \varphi \to \psi$ in E such that $pf = u \circ v$, there exists an arrow $u_v[f]\colon \varphi \to u^*\psi$ in E such that $p(u_v[f]) = v$.

$$\frac{v\colon \Lambda \to \Gamma \quad u\colon \Gamma \to \Delta \quad u \circ v\colon \Lambda \to \Delta \vdash f\colon \varphi \to \psi}{v\colon \Lambda \to \Gamma \vdash u_v[f]\colon \varphi \to u^*\psi}\ F_3$$

A functor p satisfies F_4 if for arrow $u\colon \Gamma \to \Delta$ in B and object ψ in E such that $p\psi = \Delta$, the arrow $u_{\mathbf{Id}}[\overline{u}(\psi)]\colon u^*\psi \to u^*\psi$ is equal to the identity on $u^*\psi$.

$$\frac{u\colon \Gamma \to \Delta \quad \Delta \vdash \psi}{\mathbf{Id}\colon \Gamma \to \Gamma \vdash \mathbf{Id} = u_{\mathbf{Id}}[\overline{u}(\psi)]\colon u^*\psi \to u^*\psi}\ F_4$$

A functor p satisfies F_5 if for arrow $v\colon \Lambda \to \Gamma$, $u\colon \Gamma \to \Delta$ in B and arrow f in E such that $pf = u \circ v$, the arrow $\overline{u}(\psi) \circ u_v[f]$ is equal to the arrow f.

$$\frac{v\colon \Lambda \to \Gamma \quad u\colon \Gamma \to \Delta \quad u \circ v\colon \Lambda \to \Delta \vdash f\colon \varphi \to \psi}{u \circ v\colon \Lambda \to \Delta \vdash f = \overline{u}(\psi) \circ u_v[f]\colon \varphi \to \psi}\ F_5$$

A functor p satisfies F_6 if for arrow $w\colon \Xi \to \Lambda$, $v\colon \Lambda \to \Gamma$, $u\colon \Gamma \to \Delta$ in B and arrow $g\colon \sigma \to \varphi$ and $f\colon \varphi \to \psi$ in E such that $pg = w$ and $pf = u \circ v$, the arrow $u_v[f] \circ g$ is equal to the arrow $u_{v \circ w}[f \circ g]$.

$$\frac{v\colon \Lambda \to \Gamma \quad u\colon \Gamma \to \Delta \quad u \circ v\colon \Lambda \to \Delta \vdash f\colon \varphi \to \psi \quad w\colon \Xi \to \Lambda \vdash g\colon \sigma \to \varphi}{v \circ w\colon \Xi \to \Gamma \vdash u_{v \circ w}[f \circ g] = u_v[f] \circ g\colon \sigma \to u^*\psi}\ F_6$$

Theorem 1. *A functor $p\colon E \to B$ satisfies properties F_1 through F_6 if and only if p is a cloven fibration.*

Proof (sketch). Here, we prove only that if a functor p satisfies the properties F_1 through F_6 then $\overline{u}(\psi)$ is a cartesian over u. By F_3 and F_5, for arrow $v\colon \Lambda \to \Gamma$, $u\colon \Gamma \to \Delta$ in B and arrow $f\colon \varphi \to \psi$ in E such that $pf = u \circ v$, there exists an arrow $u_v[f]\colon \varphi \to u^*\psi$ in E such that $p(u_v[f]) = v$ and $f = \overline{u}(\psi) \circ u_v[f]$. Let $g\colon \varphi \to u^*\psi$ satisfy $pg = v$ and $f = \overline{u}(\psi) \circ g$. By F_6, $u_{\mathbf{Id}}[\overline{u}(\psi)] \circ g$ is equal to $u_v[\overline{u}(\psi) \circ g]$. By $f = \overline{u}(\psi) \circ g$ and F_4, g is equal to $u_v[f]$. Therefore, $u_v[f]$ is a unique arrow such that $p(u_v[f]) = v$ and $f = \overline{u}(\psi) \circ u_v[f]$.

3 Lawvere $\mathbf{Cat}_o^{\rightarrow}$-Theories for Cloven Fibrations

In this section, we define an algebraic structure for which the models are all cloven fibrations. In the paper [6], Nishizawa and Power introduce a generalised notion of Lawvere theory, *Lawvere A-theory*. We use it as a presentation of algebraic structures. First, we show the definitions of Lawvere A-theories and models. Next, we define a Lawvere A-theory corresponding to the six properties for cloven fibrations. An arbitrary model of the Lawvere A-theory is a cloven fibration and an arbitrary cloven fibration is a model of the Lawvere A-theory.

The notion of Lawvere A-theory generalises that of classical Lawvere theory [1] in two points. First, we can enrich our theories in a category V that is locally finitely presentable as a symmetric monoidal closed category. Second, we can give arities of our theories by finitely presentable objects of a locally finitely presentable V-category A. The classical Lawvere theories are the instances where $V = A = \mathbf{Set}$.

We write A_f for a skeleton of the full sub-V-category of A given by the finitely presentable objects of A, and we let $\iota\colon A_f \to A$ denote the inclusion V-functor. Following the canonical reference for enriched categories [5], we denote the composite V-functor

$$A \xrightarrow{Y} [A^{\mathrm{op}}, V] \xrightarrow{[\iota^{\mathrm{op}}, V]} [A_f^{\mathrm{op}}, V]$$

by $\tilde{\iota}$, where Y is an enriched version of the Yoneda embedding.

Definition 6 (Lawvere A-theory). *A* Lawvere A-theory *is a small V-category L together with an identity-on-objects strict finite V-limit preserving V-functor $J\colon A_f^{\mathrm{op}} \to L$.*

The objects of L are exactly the objects of A_f^{op}; they are to be understood as generalised *arities*. The arrows of L are *operations*. (In the classical case, an arity is a finite set $n = 0, 1, \cdots, n-1$, and $f : m \to n$ is an operation taking m arguments and returning n results.)

For later use, we define some categories.

- 0 is the empty category (no objects, no arrows).
- 1 is the category with one object and one (identity) arrow.
- $\to$ is the category with two objects and just one arrow not the identity.
- $\to \cdot \to$ is the category freely generated from the following graph.

$$\cdot \xrightarrow{\mathbf{pre}} \cdot \xrightarrow{\mathbf{post}} \cdot$$

- $\to \cdot \to \cdot \to$ is the category freely generated from the following graph.

$$\cdot \xrightarrow{\mathbf{first}} \cdot \xrightarrow{\mathbf{second}} \cdot \xrightarrow{\mathbf{third}} \cdot$$

Let $\mathbf{Cat}_o^{\to}$ be the functor category from $\to$ to $\mathbf{Cat}$. We can prove that it is a locally finitely presentable category.

For later use, we define some finitely presentable objects in $\mathbf{Cat}_o^{\to}$.

- $!_1$ is the unique functor from 0 to 1.
- $!_{\to}$ is the unique functor from 0 to $\to$.
- $\mathbf{Id}_1$, $\mathbf{Id}_{\to}$, and $\mathbf{Id}_{\to\cdot\to}$ are the identity functors on 1, $\to$, and $\to \cdot \to$, respectively.
- $\mathbf{cod}$ is a functor from the category 1 to the category $\to$, which sends the unique object of 1 to the codomain of the non-identity arrow in $\to$.
- $\mathbf{comp}$ is a functor from the category $\to$ to the category $\to \cdot \to$, which sends the non-identity arrow of $\to$ to $\mathbf{post} \circ \mathbf{pre}$.
- $\mathbf{ext}$ is a functor from the category $\to \cdot \to$ to the category $\to \cdot \to \cdot \to$, which sends $\mathbf{pre}$ to $\mathbf{first}$ and sends $\mathbf{post}$ to $\mathbf{third} \circ \mathbf{second}$.

Definition 7 (Model of Lawvere A-theory). *For a Lawvere A-theory L with $J\colon A_f^{\mathrm{op}} \to L$, a* model *is an object of $\mathbf{Mod}(L)$ given by the following pullback in the category V-$\mathbf{Cat}$ of locally small V-categories.*

$$\begin{array}{ccc}
\mathbf{Mod}(L) & \longrightarrow & [L, V] \\
\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle [J,V]} \\
A & \xrightarrow[\tilde{\iota}]{} & [A_f^{\mathrm{op}}, V]
\end{array}$$

So, a model of Lawvere A-theory (L, J) is a pair of an object $a \in A$ and a V-functor $S\colon L \to V$ such that $A(\iota -, a) = S \circ J\colon A_f^{\mathrm{op}} \to V$.

Our goal is to define a Lawvere $\mathbf{Cat}_o^{\to}$-theory for which models are all cloven fibrations. First, let $\mathbf{L}$ be the freely generated Lawvere $\mathbf{Cat}_o^{\to}$-theory from $(\mathbf{Cat}_o^{\to})_f^{\mathrm{op}}$ by adding the new arrows called as follows.

$$\begin{array}{ll} \mathbf{F}_1\colon \mathbf{cod} \to \mathbf{Id}_1 & \mathbf{F}_4\colon \mathbf{cod} \to \mathbf{Id}_{\to} \\ \mathbf{F}_2\colon \mathbf{cod} \to \mathbf{Id}_{\to} & \mathbf{F}_5\colon \mathbf{comp} \to \mathbf{Id}_{\to \cdot \to} \\ \mathbf{F}_3\colon \mathbf{comp} \to \mathbf{Id}_{\to} & \mathbf{F}_6\colon \mathbf{ext} \to \mathbf{Id}_{\to \cdot \to} \end{array}$$

Let (p, S) be a model of $\mathbf{L}$. By the definition of models, p is an object $p\colon \mathbf{E} \to \mathbf{B}$ of $\mathbf{Cat}_o^{\to}$ and S is a functor $S\colon \mathbf{L} \to \mathbf{Set}$. The functor S sends the arrow $\mathbf{F}_1$ to a function $S\mathbf{F}_1\colon \mathbf{Cat}_o^{\to}(\mathbf{cod}, p) \to \mathbf{Cat}_o^{\to}(\mathbf{Id}_1, p)$. The function $S\mathbf{F}_1$ sends an element of $\mathbf{Cat}_o^{\to}(\mathbf{cod}, p)$ to an element of $\mathbf{Cat}_o^{\to}(\mathbf{Id}_1, p)$. An element of $\mathbf{Cat}_o^{\to}(\mathbf{cod}, p)$ corresponds to a pair of an arrow $u\colon \Gamma \to \Delta$ in B and an object ψ in E such that $p\psi = \Delta$. An element of $\mathbf{Cat}_o^{\to}(\mathbf{Id}_1, p)$ corresponds to a pair of an object Ξ in B and an object φ in E such that $p\varphi = \Xi$. Therefore, $S\mathbf{F}_1$ represents that for arrow $u\colon \Gamma \to \Delta$ in B and an object ψ in E such that $p\psi = \Delta$, there exist an object Ξ in B and an object φ in E such that $p\varphi = \Xi$.

$$\frac{u\colon \Gamma \to \Delta \quad \Delta \vdash \psi}{\Xi \vdash \varphi}\ \mathbf{F}_1$$

However, a model (p, S) is not always a cloven fibration. Remember that $p\colon \mathbf{E} \to \mathbf{B}$ satisfies the first property F_1 if for arrow $u\colon \Gamma \to \Delta$ in B and object ψ in E such that $p\psi = \Delta$, there exists an object $u^*\psi$ in E such that $p(u^*\psi) = \Gamma$. When $S\mathbf{F}_1$ sends $(u\colon \Gamma \to \Delta, \psi)$ to (Ξ, φ), however, Ξ is not always equal to Γ.

$$\frac{u\colon \Gamma \to \Delta \quad \Delta \vdash \psi}{\Gamma \vdash u^*\psi}\ F_1$$

Therefore, we give certain equations among arrows of the Lawvere $\mathbf{Cat}_o^{\to}$-theory $\mathbf{L}$. Arrows of $\mathbf{L}$ consist of the above new arrows $\mathbf{F}_1$ through $\mathbf{F}_6$ and arrows of $(\mathbf{Cat}_o^{\to})_f^{\mathrm{op}}$. Equations among arrows of $\mathbf{L}$ are preserved by S of all model (p, S). Therefore, they represent conditions of functions $S\mathbf{F}_1$ through $S\mathbf{F}_6$. We precisely define them in Appendix A.

Theorem 2. *There exists a Lawvere $\mathbf{Cat}_o^{\to}$-theory $\mathbf{Fb}$ for which the models are all cloven fibrations.*

4 Lawvere $\mathbf{Cat}_o^{\to}$-Theories for Fibred Structures

In this section, we extend the Lawvere $\mathbf{Cat}_o^{\to}$-theory $\mathbf{Fb}$ to make the models cloven fibrations with some fibred structures [4]. For space reasons, we show only rule-based presentation of each structures.

4.1 Fibred Terminal Object

A fibration $p\colon \mathbf{E} \to \mathbf{B}$ has *fibred terminal objects* if each fibre $\mathbf{E}_\Gamma$ has terminal objects and each reindexing functor preserves terminal objects.

In the following rules, the object 1_Γ represents a terminal object of a fibre on Γ. The arrow $!_\psi$ represents a canonical arrow from ψ to the terminal object. The rules T_3 and T_4 imply uniqueness of the arrow. The arrow $!^{-1}_{u^*1_\Delta}$ represents the inverse arrow of $!_{u^*1_\Delta}$.

$$\frac{\Gamma\colon \mathbf{base}}{\Gamma \vdash 1_\Gamma}\,T_1 \qquad \frac{\Gamma \vdash \psi}{\Gamma \vdash !_\psi\colon \psi \to 1_\Gamma}\,T_2$$

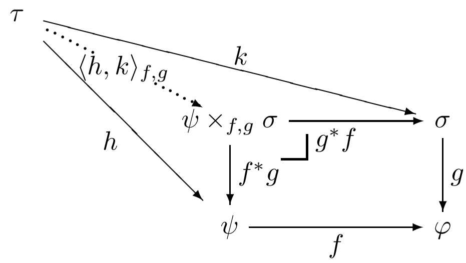
$$\frac{\Gamma\colon \mathbf{base}}{\Gamma \vdash \mathbf{Id} = !_{1_\Gamma}\colon 1_\Gamma \to 1_\Gamma}\,T_3 \qquad \frac{\Gamma \vdash f\colon \psi \to \varphi}{\Gamma \vdash !_\varphi \circ f = !_\psi\colon \psi \to 1_\Gamma}\,T_4$$

$$\frac{u\colon \Gamma \to \Delta}{\Gamma \vdash !^{-1}_{u^*1_\Delta}\colon 1_\Gamma \to u^*1_\Delta}\,T_5 \qquad \frac{u\colon \Gamma \to \Delta}{\Gamma \vdash \mathbf{Id} = !^{-1}_{u^*1_\Delta} \circ !_{u^*1_\Delta}\colon u^*1_\Delta \to u^*1_\Delta}\,T_6$$

Theorem 3. *There exists a Lawvere* $\mathbf{Cat}_o^{\to}$*-theory* $\mathbf{Fbt}$ *for which the models are all cloven fibrations that have fibred terminal objects.*

4.2 Fibred Finite Limit

A fibration $p\colon \mathbf{E} \to \mathbf{B}$ has *fibred finite limits* if each fibre $\mathbf{E}_\Gamma$ has finite limits and each reindexing functor preserves finite limits. The object $\psi \times_{f,g} \sigma$ and arrows f^*g and g^*f represent the following pullback.



The rules P_1 through P_4 represent that the pullback diagram commutes.

$$\frac{\Gamma \vdash f\colon \psi \to \varphi \quad \Gamma \vdash g\colon \sigma \to \varphi}{\Gamma \vdash \psi \times_{f,g} \sigma}\,P_1 \qquad \frac{\Gamma \vdash f\colon \psi \to \varphi \quad \Gamma \vdash g\colon \sigma \to \varphi}{\Gamma \vdash f^*g\colon \psi \times_{f,g} \sigma \to \psi}\,P_2$$

$$\frac{\Gamma \vdash f\colon \psi \to \varphi \quad \Gamma \vdash g\colon \sigma \to \varphi}{\Gamma \vdash g^*f\colon \psi \times_{f,g} \sigma \to \sigma}\,P_3 \qquad \frac{\Gamma \vdash f\colon \psi \to \varphi \quad \Gamma \vdash g\colon \sigma \to \varphi}{\Gamma \vdash f \circ f^*g = g \circ g^*f\colon \psi \times_{f,g} \sigma \to \varphi}\,P_4$$

The rules P_5 through P_7 represent existence of canonical arrow to the pullback. The rules P_8 and P_9 imply uniqueness of the arrow.

$$\frac{\Gamma \vdash f\colon \psi \to \varphi \quad \Gamma \vdash g\colon \sigma \to \varphi \quad \Gamma \vdash f \circ h = g \circ k\colon \tau \to \varphi}{\Gamma \vdash \langle h, k\rangle_{f,g}\colon \tau \to \psi \times_{f,g} \sigma}\,P_5$$

$$\frac{\Gamma \vdash f\colon \psi \to \varphi \quad \Gamma \vdash g\colon \sigma \to \varphi \quad \Gamma \vdash f \circ h = g \circ k\colon \tau \to \varphi}{\Gamma \vdash f^* g \circ \langle h, k\rangle_{f,g} = h\colon \tau \to \psi}\ P_6$$

$$\frac{\Gamma \vdash f\colon \psi \to \varphi \quad \Gamma \vdash g\colon \sigma \to \varphi \quad \Gamma \vdash f \circ h = g \circ k\colon \tau \to \varphi}{\Gamma \vdash g^* f \circ \langle h, k\rangle_{f,g} = k\colon \tau \to \sigma}\ P_7$$

$$\frac{\Gamma \vdash f\colon \psi \to \varphi \quad \Gamma \vdash g\colon \sigma \to \varphi}{\Gamma \vdash \mathbf{Id} = \langle f^* g, g^* f\rangle_{f,g}\colon \psi \times_{f,g} \sigma \to \psi \times_{f,g} \sigma}\ P_8$$

$$\frac{\Gamma \vdash f\colon \psi \to \varphi \quad \Gamma \vdash g\colon \sigma \to \varphi \quad \Gamma \vdash f \circ h = g \circ k\colon \tau \to \varphi \quad \Gamma \vdash l\colon \rho \to \tau}{\Gamma \vdash \langle h \circ l, k \circ l\rangle_{f,g} = \langle h, k\rangle_{f,g} \circ l\colon \rho \to \psi \times_{f,g} \sigma}\ P_9$$

In the rule P_{10} and P_{12}, we write $u^* f$ for $u_{\mathbf{Id}}[f \circ \overline{u}(\varphi)]\colon u^*\varphi \to u^*\psi$. It represents the arrow part of the reindexing functor u^*. In the rule P_{11} and P_{12}, we use the following abbreviation $\alpha(u, f, g)$. It represents canonical isomorphism for pullbacks preserved by reindexing functors.

$$\alpha(u, f, g) = \langle u^*(f^* g), u^*(g^* f)\rangle_{u^* f, u^* g}$$

$$\frac{u\colon \Gamma \to \Delta \quad \Delta \vdash f\colon \psi \to \varphi \quad \Delta \vdash g\colon \sigma \to \varphi}{\Gamma \vdash \alpha(u, f, g)^{-1}\colon (u^*\psi) \times_{u^* f, u^* g} (u^*\sigma) \to u^*(\psi \times_{f,g} \sigma)}\ P_{10}$$

$$\frac{u\colon \Gamma \to \Delta \quad \Delta \vdash f\colon \psi \to \varphi \quad \Delta \vdash g\colon \sigma \to \varphi}{\Gamma \vdash \mathbf{Id} = \alpha(u, f, g)^{-1} \circ \alpha(u, f, g)\colon u^*(\psi \times_{f,g} \sigma) \to u^*(\psi \times_{f,g} \sigma)}\ P_{11}$$

$$\frac{u\colon \Gamma \to \Delta \quad \Delta \vdash f\colon \psi \to \varphi \quad \Delta \vdash g\colon \sigma \to \varphi}{\Gamma \vdash \mathbf{Id} = \alpha(u, f, g) \circ \alpha(u, f, g)^{-1}\colon (u^*\psi) \times_{u^* f, u^* g} (u^*\sigma) \to (u^*\psi) \times_{u^* f, u^* g} (u^*\sigma)}\ P_{12}$$

Theorem 4. *There exists a Lawvere* $\mathbf{Cat}_o^{\to}$*-theory* **Fbfl** *for which the models are all cloven fibrations that have fibred finite limits.*

4.3 Fibred Finite Colimit

A fibration $p\colon \mathbf{E} \to \mathbf{B}$ has *fibred finite colimits* if each fibre $\mathbf{E}_\Gamma$ has finite colimits and each reindexing functor preserves finite colimits.

Theorem 5. *There exists a Lawvere* $\mathbf{Cat}_o^{\to}$*-theory* **Fbfc** *for which the models are all cloven fibrations that have fibred finite colimits.*

4.4 Cocomplete Fibration

Let $\mathbf{B}$ be a category with pullbacks. A cloven fibration $p\colon \mathbf{E} \to \mathbf{B}$ is *cocomplete* [4] if

1. p has fibred finite colimits;
2. for arrow $u\colon \Gamma \to \Delta$ in $\mathbf{B}$, the induced reindexing functor u^* has a left adjoint $\Sigma_u\colon \mathbf{E}_\Gamma \to \mathbf{E}_\Delta$;

3. the Beck-Chevalley condition holds: for pullback square

$$\begin{array}{ccc} \Lambda & \xrightarrow{\;y\;} & \Gamma \\ {\scriptstyle w}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle u} \\ \Xi & \xrightarrow[\;x\;]{} & \Delta \end{array}$$

in $\mathbf{B}$, the canonical natural transformation $\Sigma_w y^* \Rightarrow x^*\Sigma_u \colon \mathbf{E}_\Gamma \to \mathbf{E}_\Xi$ is an isomorphism.

The rules L_1 and L_2 represent the object part and the arrow part of the left adjoint Σ_u of u^*, respectively. The rules L_1 through L_4 represent that Σ_u is a functor.

$$\frac{u\colon \Gamma \to \Delta \quad \Gamma \vdash \psi}{\Delta \vdash \Sigma_u(\psi)}\ L_1 \qquad \frac{u\colon \Gamma \to \Delta \quad \Gamma \vdash f\colon \psi \to \varphi}{\Delta \vdash \Sigma_u(f)\colon \Sigma_u(\psi) \to \Sigma_u(\varphi)}\ L_2$$

$$\frac{u\colon \Gamma \to \Delta \quad \Gamma \vdash \psi}{\Delta \vdash \mathbf{Id} = \Sigma_u(\mathbf{Id})\colon \Sigma_u(\psi) \to \Sigma_u(\psi)}\ L_3$$

$$\frac{u\colon \Gamma \to \Delta \quad \Gamma \vdash f\colon \psi \to \varphi \quad \Gamma \vdash g\colon \varphi \to \sigma}{\Delta \vdash \Sigma_u(g \circ f) = \Sigma_u(g) \circ \Sigma_u(f)\colon \Sigma_u(\psi) \to \Sigma_u(\sigma)}\ L_4$$

The rule U_1 represents the φ-component $\eta_u(\varphi)$ of the unit η_u of the adjunction $\Sigma_u \dashv u^*$. The rule C_1 represents the φ-component $\epsilon_u(\varphi)$ of the counit ϵ_u of the adjunction. The rules U_2 and C_2 represent naturality of them. The rules A_1 and A_2 represent triangle conditions for the adjunction $\Sigma_u \dashv u^*$.

$$\frac{u\colon \Gamma \to \Delta \quad \Gamma \vdash \varphi}{\Gamma \vdash \eta_u(\varphi)\colon \varphi \to u^*(\Sigma_u(\varphi))}\ U_1 \qquad \frac{u\colon \Gamma \to \Delta \quad \Delta \vdash \varphi}{\Delta \vdash \epsilon_u(\varphi)\colon \Sigma_u(u^*(\varphi)) \to \varphi}\ C_1$$

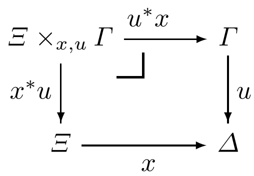
$$\frac{u\colon \Gamma \to \Delta \quad \Gamma \vdash f\colon \varphi \to \psi}{\Gamma \vdash u^*(\Sigma_u(f)) \circ \eta_u(\varphi) = \eta_u(\psi) \circ f\colon \varphi \to u^*(\Sigma_u(\psi))}\ U_2$$

$$\frac{u\colon \Gamma \to \Delta \quad \Delta \vdash f\colon \varphi \to \psi}{\Delta \vdash f \circ \epsilon_u(\varphi) = \epsilon_u(\psi) \circ \Sigma_u(u^*(f))\colon \Sigma_u(u^*(\varphi)) \to \psi}\ C_2$$

$$\frac{u\colon \Gamma \to \Delta \quad \Gamma \vdash \varphi}{\Delta \vdash \mathbf{Id} = \epsilon_u(\Sigma_u(\varphi)) \circ \Sigma_u(\eta_u(\varphi))\colon \Sigma_u(\varphi) \to \Sigma_u(\varphi)}\ A_1$$

$$\frac{u\colon \Gamma \to \Delta \quad \Delta \vdash \varphi}{\Gamma \vdash \mathbf{Id} = u^*(\epsilon_u(\varphi)) \circ \eta_u(u^*(\varphi))\colon u^*(\varphi) \to u^*(\varphi)}\ A_2$$

Similarly to fibred finite limits, we can define a Lawvere $\mathbf{Cat}_o^{\rightarrow}$-theory for which the models are all cloven fibrations $p\colon \mathbf{E} \to \mathbf{B}$ that have pullbacks in the base category $\mathbf{B}$. Therefore, we write the pullback of u and x as follows.



In the rules B_2 and B_3, we use the following abbreviations $\beta(u, x, \psi)$. It represents ψ-component of the canonical natural transformation induced by the pullback of u and x.

$$\delta_{u,x}(\Sigma_u(\psi)) = (x^*u)_{\mathbf{Id}}[x_{x^*u}[\overline{u}(\Sigma_u(\psi)) \circ \overline{u^*x}(u^*(\Sigma_u(\psi)))]]$$

$$\beta(u, x, \psi) = \epsilon_{x^*u}(x^*(\Sigma_u(\psi))) \circ \Sigma_{x^*u}(\delta_{u,x}(\Sigma_u(\psi)) \circ (u^*x)(\eta_u(\psi)))$$

$$\frac{u\colon \Gamma \to \Delta \quad x\colon \Xi \to \Delta \quad \Gamma \vdash \psi}{\Xi \vdash \beta(u, x, \psi)^{-1}\colon x^*(\Sigma_u(\psi)) \to \Sigma_{x^*u}((u^*x)^*(\psi))}\ B_1$$

$$\frac{u\colon \Gamma \to \Delta \quad x\colon \Xi \to \Delta \quad \Gamma \vdash \psi}{\Xi \vdash \mathbf{Id} = \beta(u, x, \psi)^{-1} \circ \beta(u, x, \psi)\colon \Sigma_{x^*u}((u^*x)^*(\psi)) \to \Sigma_{x^*u}((u^*x)^*(\psi))}\ B_2$$

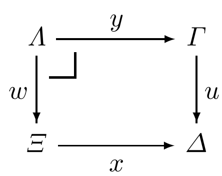
$$\frac{u\colon \Gamma \to \Delta \quad x\colon \Xi \to \Delta \quad \Gamma \vdash \psi}{\Xi \vdash \mathbf{Id} = \beta(u, x, \psi) \circ \beta(u, x, \psi)^{-1}\colon x^*(\Sigma_u(\psi)) \to x^*(\Sigma_u(\psi))}\ B_3$$

Theorem 6. *There exists a Lawvere* $\mathbf{Cat}_o^{\to}$*-theory* **Fbco** *for which the models are all cocomplete fibrations.*

4.5 Complete Fibration

Let $\mathbf{B}$ be a category with pullbacks. A fibration $p\colon \mathbf{E} \to \mathbf{B}$ is *complete* if

1. p has fibred finite limits;
2. for arrow $u\colon \Gamma \to \Delta$ in $\mathbf{B}$, the induced reindexing functor u^* has a right adjoint $\Pi_u\colon \mathbf{E}_\Gamma \to \mathbf{E}_\Delta$;
3. the Beck-Chevalley condition holds: for pullback square

   in $\mathbf{B}$, the canonical natural transformation $x^*\Pi_u \Rightarrow \Pi_w y^*\colon \mathbf{E}_\Gamma \to \mathbf{E}_\Xi$ is an isomorphism.

Theorem 7. *There exists a Lawvere* $\mathbf{Cat}_o^{\to}$*-theory* **Fbc** *for which the models are all complete fibrations.*

4.6 Fibred CCC

A fibration $p\colon \mathbf{E} \to \mathbf{B}$ is a *fibred CCC* if each fibre $\mathbf{E}_\Gamma$ is a cartesian closed category and each reindexing functor preserves finite products and exponentials.

Since fibred finite products are fibred finite limits, we can define a Lawvere $\mathbf{Cat}_o^{\to}$-theory for which the models are all cloven fibrations that have fibred finite products. For space reasons, we show only four rules for fibred binary products. The rules E_1 and E_2 represent the object part and the arrow part of the endofunctor $(-) \times \sigma$ on the fibre $\mathbf{E}_\Gamma$ for each object σ of $\mathbf{E}_\Gamma$. The rules E_3 and E_4 represent the isomorphisms for binary products preserved by reindexing functors.

$$\frac{\Gamma \vdash \psi \quad \Gamma \vdash \sigma}{\Gamma \vdash \psi \times \sigma}\,E_1 \qquad \frac{\Gamma \vdash f\colon \psi \to \varphi \quad \Gamma \vdash \sigma}{\Gamma \vdash f \times \sigma\colon \psi \times \sigma \to \varphi \times \sigma}\,E_2$$

$$\frac{u\colon \Gamma \to \Delta \quad \Delta \vdash \sigma \quad \Delta \vdash \psi}{\Gamma \vdash \mu(u, \psi, \sigma)\colon u^*(\psi \times \sigma) \to u^*\psi \times u^*\sigma}\,E_3$$

$$\frac{u\colon \Gamma \to \Delta \quad \Delta \vdash \sigma \quad \Delta \vdash \psi}{\Gamma \vdash \mu(u, \psi, \sigma)^{-1}\colon u^*\psi \times u^*\sigma \to u^*(\psi \times \sigma)}\,E_4$$

The rules E_5 and E_6 represent the object part and the arrow part of the right adjoint $[\sigma, -]$ of $(-) \times \sigma$, respectively. The rules E_5 through E_8 represent that $[\sigma, -]$ is a functor.

$$\frac{\Gamma \vdash \sigma \quad \Gamma \vdash \psi}{\Gamma \vdash [\sigma, \psi]}\,E_5 \qquad \frac{\Gamma \vdash \sigma \quad \Gamma \vdash f\colon \psi \to \varphi}{\Gamma \vdash [\sigma, f]\colon [\sigma, \psi] \to [\sigma, \varphi]}\,E_6$$

$$\frac{\Gamma \vdash \sigma \quad \Gamma \vdash \psi}{\Gamma \vdash \mathbf{Id} = [\sigma, \mathbf{Id}]\colon [\sigma, \psi] \to [\sigma, \psi]}\,E_7$$

$$\frac{\Gamma \vdash \sigma \quad \Gamma \vdash f\colon \psi \to \varphi \quad \Gamma \vdash g\colon \varphi \to \tau}{\Gamma \vdash [\sigma, g] \circ [\sigma, f] = [\sigma, g \circ f]\colon [\sigma, \psi] \to [\sigma, \tau]}\,E_8$$

The rule E_9 represents the ψ-component $\eta_\sigma(\psi)$ of the unit η_σ of the adjunction $(-) \times \sigma \dashv [\sigma, -]$. The rule E_{10} represents the ψ-component $\epsilon_\sigma(\psi)$ of the counit ϵ_σ of the adjunction. The rules E_{11} and E_{12} represent naturality of them. The rules E_{13} and E_{14} represent triangle conditions for the adjunction $(-) \times \sigma \dashv [\sigma, -]$.

$$\frac{\Gamma \vdash \sigma \quad \Gamma \vdash \psi}{\Gamma \vdash \eta_\sigma(\psi)\colon \psi \to [\sigma, \psi \times \sigma]}\,E_9 \qquad \frac{\Gamma \vdash \sigma \quad \Gamma \vdash \psi}{\Gamma \vdash \epsilon_\sigma(\psi)\colon [\sigma, \psi] \times \sigma \to \psi}\,E_{10}$$

$$\frac{\Gamma \vdash \sigma \quad \Gamma \vdash f\colon \psi \to \varphi}{\Gamma \vdash \eta_\sigma(\varphi) \circ f = [\sigma, f \times \sigma] \circ \eta_\sigma(\psi)\colon \psi \to [\sigma, \varphi \times \sigma]}\,E_{11}$$

$$\frac{\Gamma \vdash \sigma \quad \Gamma \vdash f\colon \psi \to \varphi}{\Gamma \vdash f \circ \epsilon_\sigma(\psi) = \epsilon_\sigma(\varphi) \circ ([\sigma, f] \times \sigma)\colon [\sigma, \psi] \times \sigma \to \varphi}\,E_{12}$$

$$\frac{\Gamma \vdash \sigma \quad \Gamma \vdash \psi}{\Gamma \vdash \mathbf{Id} = [\sigma, \epsilon_\sigma(\psi)] \circ \eta_\sigma([\sigma, \psi])\colon [\sigma, \psi] \to [\sigma, \psi]}\,E_{13}$$

$$\frac{\Gamma \vdash \sigma \quad \Gamma \vdash \psi}{\Gamma \vdash \mathbf{Id} = \epsilon_\sigma(\psi \times \sigma) \circ (\eta_\sigma(\psi) \times \sigma) \colon \psi \times \sigma \to \psi \times \sigma}\ E_{14}$$

In the rule E_{16} and E_{17}, we use the following abbreviation $\nu(u, \sigma, \psi)$. It represents canonical isomorphism for exponents preserved by reindexing functors.

$$\nu(u, \sigma, \psi) = [u^*\sigma, u^*\epsilon_\sigma(\psi)] \circ [u^*\sigma, \mu(u, [\sigma, \psi], \sigma)^{-1}] \circ \eta_{u^*\sigma}(u^*[\sigma, \psi])$$

$$\frac{u \colon \Gamma \to \Delta \quad \Delta \vdash \sigma \quad \Delta \vdash \psi}{\Gamma \vdash \nu(u, \sigma, \psi)^{-1} \colon [u^*\sigma, u^*\psi] \to u^*[\sigma, \psi]}\ E_{15}$$

$$\frac{u \colon \Gamma \to \Delta \quad \Delta \vdash \sigma \quad \Delta \vdash \psi}{\Gamma \vdash \mathbf{Id} = \nu(u, \sigma, \psi)^{-1} \circ \nu(u, \sigma, \psi) \colon u^*[\sigma, \psi] \to u^*[\sigma, \psi]}\ E_{16}$$

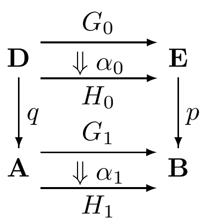
$$\frac{u \colon \Gamma \to \Delta \quad \Delta \vdash \sigma \quad \Delta \vdash \psi}{\Gamma \vdash \mathbf{Id} = \nu(u, \sigma, \psi) \circ \nu(u, \sigma, \psi)^{-1} \colon [u^*\sigma, u^*\psi] \to [u^*\sigma, u^*\psi]}\ E_{17}$$

Theorem 8. *There exists a Lawvere* $\mathbf{Cat}_o^{\to}$*-theory* **Fbccc** *for which the models are all fibred CCCs.*

5 Enrichability of Lawvere $\mathbf{Cat}_o^{\to}$-Theories

In this section, we analyse the Lawvere $\mathbf{Cat}_o^{\to}$-Theories in the previous section, in the point of view of enriched category theory [5]. We extend some of them to **Cat**-enriched one. The models and morphisms remain the same, but we also have 2-cells. On the one hand, the Lawvere $\mathbf{Cat}_o^{\to}$-Theory for cocomplete fibrations can be **Cat**-enriched. On the other hand, we expect that some fibred structure can not be **Cat**-enriched, for example, complete fibrations and fibred CCCs.

A 2-category (i.e., **Cat**-category) $\mathbf{Cat}^{\to}$ has the same objects and arrows as $\mathbf{Cat}_o^{\to}$. For objects $q \colon \mathbf{D} \to \mathbf{A}$, $p \colon \mathbf{E} \to \mathbf{B}$, an arrow $G \colon q \to p$ consists of a functor $G_0 \colon \mathbf{D} \to \mathbf{E}$ and a functor $G_1 \colon \mathbf{A} \to \mathbf{B}$ such that $p \circ G_0 = G_1 \circ q$. For arrows $G, H \colon q \to p$, a 2-cell $\alpha \colon G \Rightarrow H$ consists of a natural transformation $\alpha_0 \colon G_0 \Rightarrow H_0$ and a natural transformation $\alpha_1 \colon G_1 \Rightarrow H_1$ such that $p\alpha_0 = \alpha_1 q$. We can prove that $\mathbf{Cat}^{\to}$ is a locally finitely presentable 2-category.



For each fibred structures, we extend the **Set**-enriched Lawvere $\mathbf{Cat}_o^{\to}$-theories to the **Cat**-enriched Lawvere $\mathbf{Cat}^{\to}$-theories if possible.

Theorem 9 (Cloven Fibration). *Let* **Fb2** *be the Lawvere* $\mathbf{Cat}^{\rightarrow}$*-theory freely generated from* $(\mathbf{Cat}^{\rightarrow})_f^{\mathrm{op}}$ *by adding the same new arrows and equations as* **Fb**. *There exists a bijection between the class of all models for* **Fb2** *and the class of all models for* **Fb**.

Proof (sketch). Let $\mathbf{ob}\colon \mathbf{Cat} \to \mathbf{Set}$ be the functor that sends a category to the set of the objects. If (p, T) is a model of **Fb2**, then $(p, \mathbf{ob} \circ T)$ is a model of **Fb**.

Conversely, we prove that for model (p, S) of **Fb**, there exists a unique model (p, T) of **Fb2** such that $\mathbf{ob} \circ T = S$. Here we show that $T\mathbf{F}_1$ is uniquely determined by a model of **Fb**. Take an arrow in the category $\mathbf{Cat}^{\rightarrow}(\mathbf{cod}, p)$ as follows.

$$\begin{array}{ccccc} \Gamma & \xrightarrow{u} & \Delta & \qquad & \psi \\ \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \delta = p\pi} & & \downarrow{\scriptstyle \pi} \\ \Gamma' & \xrightarrow{u'} & \Delta' & & \psi' \end{array}$$

The object part of the functor $T\mathbf{F}_1$ is determined by $S\mathbf{F}_1$. We define an arrow part of the functor $T\mathbf{F}_1$ as follows. Then, $T\mathbf{F}_1$ becomes a functor and (p, T) is a unique model of **Fb2** such that $\mathbf{ob} \circ T = S$.

$$\begin{array}{c} u^*\psi \\ \downarrow{\scriptstyle u'_\gamma[\pi \circ \overline{u}(\psi)]} \\ (u')^*\psi' \end{array}$$

Theorem 10 (Fibred Terminal Object). *Let* **Fbt2** *be the Lawvere* $\mathbf{Cat}^{\rightarrow}$*-theory freely generated from* $(\mathbf{Cat}^{\rightarrow})_f^{\mathrm{op}}$ *by adding the same new arrows and equations as* **Fbt**. *There exists a bijection between the class of all models for* **Fbt2** *and the class of all models for* **Fbt**.

Proof (sketch). The functor for T_1 is uniquely determined so as to send an arrow $\gamma\colon \Gamma \to \Gamma'$ to the following arrow.

$$1_\Gamma \xrightarrow{!^{-1}_{\gamma^* 1_{\Gamma'}}} \gamma^* 1_{\Gamma'} \xrightarrow{\overline{\gamma}(1_{\Gamma'})} 1_{\Gamma'}$$

Theorem 11 (Fibred Finite Limit). *Let* **Fbfl2** *be the Lawvere* $\mathbf{Cat}^{\rightarrow}$*-theory freely generated from* $(\mathbf{Cat}^{\rightarrow})_f^{\mathrm{op}}$ *by adding the same new arrows and equations as* **Fbfl**. *There exists a bijection between the class of all models for* **Fbfl2** *and the class of all models for* **Fbfl**.

Proof (sketch). The functor for P_1 is uniquely determined so as to send arrows

$$\begin{array}{ccccccccc} \psi & \xrightarrow{f} & \varphi & \qquad & \sigma & \xrightarrow{g} & \varphi & \qquad & \Gamma \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle r} & & \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle r} & & \downarrow{\scriptstyle \gamma = pq = pr = ps} \\ \psi' & \xrightarrow{f'} & \varphi' & & \sigma' & \xrightarrow{g'} & \varphi' & & \Gamma' \end{array}$$

to the following arrow (put $\iota = \langle \gamma_{\mathbf{Id}}[q \circ f^* g], \gamma_{\mathbf{Id}}[s \circ g^* f] \rangle_{\gamma^* f', \gamma^* g'}$).

$$\psi \times_{f,g} \sigma \xrightarrow{\iota} (\gamma^*\psi') \times_{\gamma^* f', \gamma^* g'} (\gamma^* \sigma') \xrightarrow{\alpha(\gamma,\psi',\sigma')^{-1}} \gamma^*(\psi' \times_{f',g'} \sigma') \xrightarrow{\overline{\gamma}(\psi' \times_{f',g'} \sigma')} \psi' \times_{f',g'} \sigma'$$

Theorem 12 (Fibred Finite Colimit). *Let* **Fbfc2** *be the Lawvere* $\mathbf{Cat}^{\rightarrow}$*-theory freely generated from* $(\mathbf{Cat}^{\rightarrow})_f^{\mathrm{op}}$ *by adding the same new arrows and equations as* **Fbfc**. *There exists a bijection between the class of all models for* **Fbfc2** *and the class of all models for* **Fbfc**.

Theorem 13 (Cocomplete Fibration). *Let* **Fbco2** *be the Lawvere* $\mathbf{Cat}^{\rightarrow}$*-theory freely generated from* $(\mathbf{Cat}^{\rightarrow})_f^{\mathrm{op}}$ *by adding the same new arrows and equations as* **Fbco**. *There exists a bijection between the class of all models for* **Fbco2** *and the class of all models for* **Fbco**.

Proof (sketch). The functor for L_1 is uniquely determined so as to send arrows

$$\begin{array}{ccccc} \Gamma & \xrightarrow{u} & \Delta & \qquad & \psi \\ \downarrow{\scriptstyle \gamma = p\pi} & & \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \pi} \\ \Gamma' & \xrightarrow{u'} & \Delta' & & \psi' \end{array}$$

to the arrow determined by the following correspondence.

$$\begin{array}{rclcc} \Sigma_u(\psi) & \longrightarrow & \Sigma_{u'}(\psi') & \text{over} & \delta \\ \hline \Sigma_u(\psi) & \longrightarrow & \delta^* \Sigma_{u'}(\psi') & \text{over} & \mathbf{Id}_\Delta \\ \hline \psi & \longrightarrow & u^* \delta^* \Sigma_{u'}(\psi') & \text{over} & \mathbf{Id}_\Gamma \\ \hline \psi & \longrightarrow & \gamma^*(u')^* \Sigma_{u'}(\psi') & \text{over} & \mathbf{Id}_\Gamma \\ \hline \psi \xrightarrow[\pi]{} \psi' & \xrightarrow[\eta_{u'}(\psi')]{} & (u')^* \Sigma_{u'}(\psi') & \text{over} & \gamma \end{array}$$

Next, we try to extend **Fbccc** to **Cat**-enriched one. We must define a functor for E_5 that sends arrows

$$\begin{array}{ccccc} \Gamma & \qquad & \sigma & \qquad & \psi \\ \downarrow{\scriptstyle \gamma = ps = p\pi} & & \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle \pi} \\ \Gamma' & & \sigma' & & \psi' \end{array}$$

to an arrow from $[\sigma, \psi]$ to $[\sigma', \psi']$. Since the algebraic structure for CCC on **Cat** can not be **Cat**-enriched[2], however, we expect there are no such functors. Therefore, we expect there are no **Cat**-enriched Lawvere $\mathbf{Cat}^{\rightarrow}$-theories for which the models are all fibred CCCs. Similarly, we expect there are no **Cat**-enriched Lawvere $\mathbf{Cat}^{\rightarrow}$-theories for which the models are all complete fibrations.

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A Lawvere $\mathbf{Cat}_o^{\rightarrow}$-theory Fb

In this appendix, we give a precise definition of Lawvere $\mathbf{Cat}_o^{\rightarrow}$-theory **Fb**.

First, we define arrows $\mathbf{G}_1$ through $\mathbf{G}_{18}$ in $(\mathbf{Cat}_o^{\rightarrow})_f^{\mathrm{op}}$ by showing functions $S\mathbf{G}_1$ through $S\mathbf{G}_{18}$ for a model (p, S) of a Lawvere $\mathbf{Cat}_o^{\rightarrow}$-theory (L, J). By the definition of models, all arrow G in $(\mathbf{Cat}_o^{\rightarrow})_f^{\mathrm{op}}$ satisfies $SG = SJG = \mathbf{Cat}_o^{\rightarrow}(\iota G, p)$.

There exists $\mathbf{G}_1 \colon \mathbf{cod} \to !_{\rightarrow}$ such that for arrow $u \colon \Gamma \to \Delta$ in B and object ψ in E such that $p\psi = \Delta$, the function $S\mathbf{G}_1$ returns u.

$$\frac{u\colon \Gamma \to \Delta \quad \Delta \vdash \psi}{u\colon \Gamma \to \Delta}\ \mathbf{G}_1$$

There exists $\mathbf{G}_2 \colon !_{\rightarrow} \to !_1$ such that for arrow $u \colon \Gamma \to \Delta$ in B, the function $S\mathbf{G}_2$ returns Γ.

$$\frac{u\colon \Gamma \to \Delta}{\Gamma\colon \mathbf{base}}\ \mathbf{G}_2$$

There exists $\mathbf{G}_3 \colon \mathbf{Id}_1 \to !_1$ such that for object Δ in B and object ψ in E such that $p\psi = \Delta$, the function $S\mathbf{G}_3$ returns Δ.

$$\frac{\Delta \vdash \psi}{\Delta\colon \mathbf{base}}\ \mathbf{G}_3$$

There exists $\mathbf{G}_4 \colon \mathbf{cod} \to \mathbf{Id}_1$ such that for arrow $u \colon \Gamma \to \Delta$ in B and object ψ in E such that $p\psi = \Delta$, the function $S\mathbf{G}_4$ returns Δ and ψ.

$$\frac{u\colon \Gamma \to \Delta \quad \Delta \vdash \psi}{\Delta \vdash \psi}\ \mathbf{G}_4$$

There exists $\mathbf{G}_5\colon \mathbf{Id}_{\to} \to \mathbf{Id}_1$ such that for arrow $u\colon \Gamma \to \Delta$ in B and arrow $f\colon \varphi \to \psi$ in E such that $pf = u$, the function $S\mathbf{G}_5$ returns Δ and ψ.

$$\frac{u\colon \Gamma \to \Delta \vdash f\colon \varphi \to \psi}{\Delta \vdash \psi}\ \mathbf{G}_5$$

There exists $\mathbf{G}_6\colon \mathbf{Id}_{\to} \to !_{\to}$ such that for arrow $u\colon \Gamma \to \Delta$ in B and arrow $f\colon \varphi \to \psi$ in E such that $pf = u$, the function $S\mathbf{G}_6$ returns u.

$$\frac{u\colon \Gamma \to \Delta \vdash f\colon \varphi \to \psi}{u\colon \Gamma \to \Delta}\ \mathbf{G}_6$$

There exists $\mathbf{G}_7\colon \mathbf{comp} \to \mathbf{Id}_{\to}$ such that for arrow $v\colon \Lambda \to \Gamma$, $u\colon \Gamma \to \Delta$ in B and arrow $f\colon \varphi \to \psi$ in E such that $pf = u \circ v$, the function $S\mathbf{G}_7$ returns f and $u \circ v$.

$$\frac{v\colon \Lambda \to \Gamma \quad u\colon \Gamma \to \Delta \quad u \circ v\colon \Lambda \to \Delta \vdash f\colon \varphi \to \psi}{u \circ v\colon \Lambda \to \Delta \vdash f\colon \varphi \to \psi}\ \mathbf{G}_7$$

There exists $\mathbf{G}_8\colon \mathbf{Id}_{\to} \to \mathbf{Id}_1$ such that for arrow $u\colon \Gamma \to \Delta$ in B and arrow $f\colon \varphi \to \psi$ in E such that $pf = u$, the function $S\mathbf{G}_8$ returns Γ and φ.

$$\frac{u\colon \Gamma \to \Delta \vdash f\colon \varphi \to \psi}{\Gamma \vdash \varphi}\ \mathbf{G}_8$$

There exists $\mathbf{G}_9\colon \mathbf{comp} \to \mathbf{cod}$ such that for arrow $v\colon \Lambda \to \Gamma$, $u\colon \Gamma \to \Delta$ in B and arrow $f\colon \varphi \to \psi$ in E such that $pf = u \circ v$, the function $S\mathbf{G}_9$ returns u and ψ.

$$\frac{v\colon \Lambda \to \Gamma \quad u\colon \Gamma \to \Delta \quad u \circ v\colon \Lambda \to \Delta \vdash f\colon \varphi \to \psi}{u\colon \Gamma \to \Delta \quad \Delta \vdash \psi}\ \mathbf{G}_9$$

There exists $\mathbf{G}_{10}\colon \mathbf{Id}_1 \to \mathbf{Id}_{\to}$ such that for object Δ in B and object ψ in E such that $p\psi = \Delta$, the function $S\mathbf{G}_{10}$ returns $\mathbf{Id}\colon \Delta \to \Delta$ and $\mathbf{Id}\colon \psi \to \psi$.

$$\frac{\Delta \vdash \psi}{\mathbf{Id}\colon \Delta \to \Delta \vdash \mathbf{Id}\colon \psi \to \psi}\ \mathbf{G}_{10}$$

There exists $\mathbf{G}_{11}\colon \mathbf{Id}_{\to} \to \mathbf{comp}$ such that for arrow $u\colon \Gamma \to \Delta$ in B and arrow $f\colon \varphi \to \psi$ in E such that $pf = u$, the function $S\mathbf{G}_{11}$ returns u, f, and $\mathbf{Id}\colon \Gamma \to \Gamma$.

$$\frac{u\colon \Gamma \to \Delta \vdash f\colon \varphi \to \psi}{\mathbf{Id}\colon \Gamma \to \Gamma \quad u\colon \Gamma \to \Delta \quad u \circ \mathbf{Id}\colon \Gamma \to \Delta \vdash f\colon \varphi \to \psi}\ \mathbf{G}_{11}$$

There exists $\mathbf{G}_{12}\colon \mathbf{Id}_{\to \cdot \to} \to \mathbf{Id}_{\to}$ such that for arrow $v\colon \Lambda \to \Gamma$, $u\colon \Gamma \to \Delta$ in B and arrow $g\colon \sigma \to \varphi$, $f\colon \varphi \to \psi$ in E such that $pg = v$ and $pf = u$, the function $S\mathbf{G}_{12}$ returns $f \circ g$ and $u \circ v$.

$$\frac{v\colon \Lambda \to \Gamma \vdash g\colon \sigma \to \varphi \quad u\colon \Gamma \to \Delta \vdash f\colon \varphi \to \psi}{u \circ v\colon \Lambda \to \Delta \vdash f \circ g\colon \sigma \to \psi}\ \mathbf{G}_{12}$$

There exists $\mathbf{G}_{13}\colon \mathbf{Id}_{\to,\to} \to \mathbf{Id}_{\to}$ such that for arrow $v\colon \Lambda \to \Gamma$, $u\colon \Gamma \to \Delta$ in B and arrow $g\colon \sigma \to \varphi$, $f\colon \varphi \to \psi$ in E such that $pg = v$ and $pf = u$, the function $S\mathbf{G}_{13}$ returns f and u.

$$\frac{v\colon \Lambda \to \Gamma \vdash g\colon \sigma \to \varphi \quad u\colon \Gamma \to \Delta \vdash f\colon \varphi \to \psi}{u\colon \Gamma \to \Delta \vdash f\colon \varphi \to \psi}\ \mathbf{G}_{13}$$

There exists $\mathbf{G}_{14}\colon \mathbf{Id}_{\to,\to} \to \mathbf{Id}_{\to}$ such that for arrow $v\colon \Lambda \to \Gamma$, $u\colon \Gamma \to \Delta$ in B and arrow $g\colon \sigma \to \varphi$, $f\colon \varphi \to \psi$ in E such that $pg = v$ and $pf = u$, the function $S\mathbf{G}_{14}$ returns g and v.

$$\frac{v\colon \Lambda \to \Gamma \vdash g\colon \sigma \to \varphi \quad u\colon \Gamma \to \Delta \vdash f\colon \varphi \to \psi}{v\colon \Lambda \to \Gamma \vdash g\colon \sigma \to \varphi}\ \mathbf{G}_{14}$$

There exists $\mathbf{G}_{15}\colon \mathbf{ext} \to \mathbf{Id}_{\to,\to}$ such that for arrow $w\colon \Xi \to \Lambda$, $v\colon \Lambda \to \Gamma$, $u\colon \Gamma \to \Delta$ in B and arrow $g\colon \sigma \to \varphi$ and $f\colon \varphi \to \psi$ in E such that $pg = w$ and $pf = u \circ v$, the function $S\mathbf{G}_{15}$ returns w, $u \circ v$, g, and f.

$$\frac{v\colon \Lambda \to \Gamma \quad u\colon \Gamma \to \Delta \quad u \circ v\colon \Lambda \to \Delta \vdash f\colon \varphi \to \psi \quad w\colon \Xi \to \Lambda \vdash g\colon \sigma \to \varphi}{u \circ v\colon \Lambda \to \Delta \vdash f\colon \varphi \to \psi \quad w\colon \Xi \to \Lambda \vdash g\colon \sigma \to \varphi}\ \mathbf{G}_{15}$$

There exists $\mathbf{G}_{16}\colon \mathbf{ext} \to \mathbf{comp}$ such that for arrow $w\colon \Xi \to \Lambda$, $v\colon \Lambda \to \Gamma$, $u\colon \Gamma \to \Delta$ in B and arrow $g\colon \sigma \to \varphi$ and $f\colon \varphi \to \psi$ in E such that $pg = w$ and $pf = u \circ v$, the function $S\mathbf{G}_{16}$ returns u, v, and f.

$$\frac{v\colon \Lambda \to \Gamma \quad u\colon \Gamma \to \Delta \quad u \circ v\colon \Lambda \to \Delta \vdash f\colon \varphi \to \psi \quad w\colon \Xi \to \Lambda \vdash g\colon \sigma \to \varphi}{v\colon \Lambda \to \Gamma \quad u\colon \Gamma \to \Delta \quad u \circ v\colon \Lambda \to \Delta \vdash f\colon \varphi \to \psi}\ \mathbf{G}_{16}$$

There exists $\mathbf{G}_{17}\colon \mathbf{ext} \to \mathbf{comp}$ such that for arrow $w\colon \Xi \to \Lambda$, $v\colon \Lambda \to \Gamma$, $u\colon \Gamma \to \Delta$ in B and arrow $g\colon \sigma \to \varphi$ and $f\colon \varphi \to \psi$ in E such that $pg = w$ and $pf = u \circ v$, the function $S\mathbf{G}_{17}$ returns u, $v \circ w$, and $f \circ g$.

$$\frac{v\colon \Lambda \to \Gamma \quad u\colon \Gamma \to \Delta \quad u \circ v\colon \Lambda \to \Delta \vdash f\colon \varphi \to \psi \quad w\colon \Xi \to \Lambda \vdash g\colon \sigma \to \varphi}{v \circ w\colon \Xi \to \Gamma \quad u\colon \Gamma \to \Delta \quad u \circ v \circ w\colon \Xi \to \Delta \vdash f \circ g\colon \sigma \to \psi}\ \mathbf{G}_{17}$$

There exists $\mathbf{G}_{18}\colon \mathbf{comp} \to !_{\to}$ such that for arrow $v\colon \Lambda \to \Gamma$, $u\colon \Gamma \to \Delta$ in B and arrow $f\colon \varphi \to \psi$ in E such that $pf = u \circ v$, the function $S\mathbf{G}_{18}$ returns v.

$$\frac{v\colon \Lambda \to \Gamma \quad u\colon \Gamma \to \Delta \quad u \circ v\colon \Lambda \to \Delta \vdash f\colon \varphi \to \psi}{v\colon \Lambda \to \Gamma}\ \mathbf{G}_{18}$$

Let $\mathbf{Fb}$ be the freely generated Lawvere $\mathbf{Cat}_o^{\to}$-theory from $\mathbf{L}$ subject to the following equations.

$$\begin{aligned}
\mathbf{G}_3 \circ \mathbf{F}_1 &= \mathbf{G}_2 \circ \mathbf{G}_1 \\
\mathbf{G}_5 \circ \mathbf{F}_2 &= \mathbf{G}_4 \\
\mathbf{G}_6 \circ \mathbf{F}_2 &= \mathbf{G}_1 \\
\mathbf{G}_8 \circ \mathbf{F}_2 &= \mathbf{F}_1 \\
\mathbf{G}_5 \circ \mathbf{F}_3 &= \mathbf{F}_1 \circ \mathbf{G}_9 \\
\mathbf{G}_6 \circ \mathbf{F}_3 &= \mathbf{G}_{18} \\
\mathbf{G}_8 \circ \mathbf{F}_3 &= \mathbf{G}_8 \circ \mathbf{G}_7
\end{aligned}
\qquad
\begin{aligned}
\mathbf{F}_4 &= \mathbf{G}_{10} \circ \mathbf{F}_1 \\
\mathbf{F}_4 &= \mathbf{F}_3 \circ \mathbf{G}_{11} \circ \mathbf{F}_2 \\
\mathbf{G}_{12} \circ \mathbf{F}_5 &= \mathbf{G}_7 \\
\mathbf{G}_{13} \circ \mathbf{F}_5 &= \mathbf{F}_2 \circ \mathbf{G}_9 \\
\mathbf{G}_{14} \circ \mathbf{F}_5 &= \mathbf{F}_3 \\
\mathbf{G}_{12} \circ \mathbf{F}_6 &= \mathbf{F}_3 \circ \mathbf{G}_{17} \\
\mathbf{G}_{13} \circ \mathbf{F}_6 &= \mathbf{F}_3 \circ \mathbf{G}_{16} \\
\mathbf{G}_{14} \circ \mathbf{F}_6 &= \mathbf{G}_{14} \circ \mathbf{G}_{15}
\end{aligned}$$

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