A free construction of Kleene algebras with tests

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Abstract

In this paper we define Kleene algebra with tests in a slightly more general way than Kozen's definition. Then we give an explicit construction of the free Kleene algebra with tests generated by a pair of sets. Moreover we show that our free Kleene algebra with tests is isomorphic to Kozen and Smith's Kleene algebra with tests if their construction available, that is, a generator of the Boolean algebra is finite. Finally, we show that an infinitely-generated free Kleene algebra with tests in the sense of Kozen can be obtained from our Free algebra.

1 Introduction

Kozen [6] defined a Kleene algebra with tests to be a Kleene algebra with an embedded Boolean algebra. The starting point of this paper is the observation that the important point of this notion is not the subset property but the fact that their underlying idempotent semiring structure is shared. Due to this observation, we define a Kleene algebra with tests as a triple consisting of a Boolean algebra, a Kleene algebra, and a function from the carrier set of the Boolean algebra to the carrier of the Kleene algebra which preserves their underlying idempotent semiring structures. So the category **KAT** of our Kleene algebras with tests is the comma category (U_{BI}, U_{KI}) of the functor U_{BI} from the category **Bool** of Boolean algebras to the category **ISR** of idempotent semirings and the functor U_{KI} from the category **Kleene** of Kleene algebra to **ISR**.

Kozen and Smith [7] showed the existence of the free Kleene algebra with tests generated by a pair of finite sets through a point-wise construction of the set of languages of guarded strings. Furusawa and Kinoshita [3, 4] showed the existence of the free algebra generated by a pair of sets using the technique of finite limit sketches instead of showing an explicit construction. These two results were established on slightly different definitions, and the lack of an explicit construction in the latter result prevents us from comparing these two.

*Part of this work is done during the author is visiting at McMaster University by Japan Society for the Promotion of Science (JSPS) Bilateral Programs for Science Exchanges.

In this paper, we give an explicit construction of the free Kleene algebra with tests generated by a pair of sets with using adjunctions. Though Kozen and Smith's construction requires finiteness of generators, our construction never requires it. Moreover, our construction makes it possible to reach to the free algebras directly, where, of course, the definition of Kleene algebras with tests is not Kozen's but ours. So we also show that our free algebra and Kozen and Smith's are isomorphic whenever Kozen and Smith's construction is available. Finally, we show that an infinitely-generated free Kleene algebra with tests in the sense of Kozen is obtained from our Free algebra.

2 Kleene algebras

In this section we recall some basic of Kleene algebras [5] and related structures. [2] contains several examples of Kleene algebras. For basic notions of category theory we refer to [1, 8].

Definition 2.1 (Kleene algebra) A *Kleene algebra* is a set K equipped with nullary operators 0, 1 and binary operators $+$, $\cdot$, and a unary operator *, where the tuple $(K, +, \cdot, 0, 1)$ is an idempotent semiring and these data satisfy the following:

$$\begin{aligned} 1 + (p \cdot p^*) &= p^* \\ 1 + (p^* \cdot p) &= p^* \\ p \cdot r \leq r &\implies p^* \cdot r \leq r \\ r \cdot p \leq r &\implies r \cdot p^* \leq r \end{aligned}$$

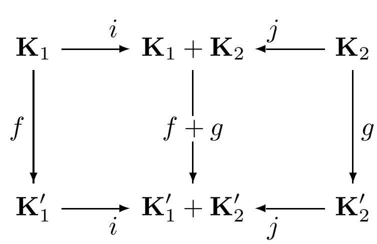
where $\leq$ refers to the natural partial order

$$p \leq q \overset{\text{def}}{\iff} p + q = q \quad .$$

A Kleene algebra will be called *trivial* if $0 = 1$, otherwise, called *non-trivial.* The category of Kleene algebras and homomorphisms between them will be denoted by **Kleene**.

Remark 2.2 Kleene *has binary coproducts.*

For two Kleene algebras $\mathbf{K}_1$ and $\mathbf{K}_2$, $\mathbf{K}_1 + \mathbf{K}_2$ denotes a coproduct of $\mathbf{K}_1$ and $\mathbf{K}_2$. For two Kleene algebra homomorphisms $f\colon \mathbf{K}_1 \to \mathbf{K}_2$ and $g\colon \mathbf{K}'_1 \to \mathbf{K}'_2$, $f + g$ denotes a unique Kleene algebra homomorphism such that the following two diagrams commutes.



where i and j are injections of coproduct $\mathbf{K}_1 + \mathbf{K}_2$, and i' and j' so are of $\mathbf{K}'_1 + \mathbf{K}'_2$.

The injections of a coproduct in **Kleene** are not always one-to-one. Trivial Kleene algebras have only one element since, for each a,

$$a = a \cdot 1 = a \cdot 0 = 0 \ .$$

For each Kleene algebra K, there exists a unique Kleene algebra homomorphism from K to trivial one. From a trivial Kleene algebra, there exists a Kleene algebra homomorphism if the target is also trivial one. So, coproduct of a trivial Kleene algebra and a non-trivial one is a trivial one again. Then, we have an injection which is not one-to-one. This example is due to Wolfram Kahl.

A Kleene algebra $\mathbf{K}$ is called *integral* if it has no zero divisors, that is,

$$a \neq 0 \ \wedge \ b \neq 0 \implies a \cdot b \neq 0$$

holds for all $a, b \in K$. This notion is introduced in [2].

Proposition 2.3 *Let* $\mathbf{J} = (J, +_J, \cdot_J, {}^{*_J}, 0_J, 1_J)$ *and* $\mathbf{K} = (K, +_K, \cdot_K, {}^{*_K}, 0_K, 1_K)$ *be non-trivial Kleene algebras. If* $\mathbf{K}$ *is integral, then the following holds.*

(i) The mapping $f\colon K \to J$ *defined to be* $f(a) = 0_J$ *if* $a = 0_K$*, and otherwise* $f(a) = 1_J$*, is a Kleene algebra homomorphism.*

(ii) The first injection $j\colon \mathbf{J} \to \mathbf{J} + \mathbf{K}$ *is one-to-one.*

Proof. Since $\mathbf{K}$ is non-trivial, we possibly have a Kleene algebra homomorphisms from $\mathbf{K}$ to non-trivial one. For each $a, b \in K$, if $a \neq 0_K$ and $b \neq 0_K$, $a +_K b \neq 0_K$ and $a \cdot_K b \neq 0_K$ since $\mathbf{K}$ is integral. So, *(i)* follows from

$$\begin{array}{l} f(a) +_J f(b) = 1_J +_J 1_J = 1_J = f(a +_K b) \\ f(a) \cdot_J f(b) = 1_J \cdot_J 1_J = 1_J = f(a \cdot_K b) \\ f(a)^{*_J} = 1_J^{*_J} = 1_J = f(a^{*_K}) \ . \end{array}$$

(ii) will be proved using f given in *(i)*. Take $\mathrm{id}_{\mathbf{J}}$ and f, then a unique intermediating arrow $h\colon \mathbf{J} + \mathbf{K} \to \mathbf{J}$ with respect to them exists. By the definition of coproducts, h satisfies $\mathrm{id}_{\mathbf{J}} = h \circ j$. Thus j is one-to-one.

Set, **ISR**, and **Bool** denote the categories of sets and functions, idempotent semirings and their homomorphisms, and Boolean algebras and their homomorphisms, respectively. $U_K\colon$ **Kleene** $\to$ **Set** denotes the forgetful functor which takes a Kleene algebra to its carrier set. The functor U_K is decomposed by functors $U_{KI}\colon$ **Kleene** $\to$ **ISR** and $U_I\colon$ **ISR** $\to$ **Set**, where $U_{KI}(\mathbf{K})$ is an idempotent semiring obtained by forgetting $*$ operator and U_I takes an idempotent semiring to its carrier set. These two functors U_{KI} and U_I have left adjoints F_{IK} and F_I respectively. $F_K \stackrel{\mathrm{def}}{=} F_{IK} \circ F_K$ is a left adjoint to U_K.

Remark 2.4 ($\mathbf{Reg}(\Sigma)$ [5]) *For a set Σ, $\mathbf{K}_\Sigma$ denotes the Kleene algebra consisting of the set $\mathbf{Reg}(\Sigma)$ of regular sets over Σ together with the standard operations on regular sets. Clearly, $\mathbf{K}_\Sigma$ is integral. Moreover, it is known that $\mathbf{K}_\Sigma \cong F_K(\Sigma)$.*

Since we consider Kleene algebras with tests in this paper, Boolean algebras are also important. We denote the forgetful functor which takes a Boolean algebra to its carrier set by $U_B \colon \mathbf{Bool} \to \mathbf{Set}$. U_B satisfies similar properties to U_K together with U_I. We denote the forgetful functor from $\mathbf{Bool}$ to $\mathbf{ISR}$ and its left adjoint by U_{BI} and F_{IB} respectively. $F_B \stackrel{\text{def}}{=} F_{IB} \circ F_I$ is a left adjoint to U_B.

The situation we state above is as follows:

$$
\begin{array}{ccccc}
\mathbf{Bool} & \underset{U_{BI}}{\overset{F_{IB}}{\rightleftarrows}} & \mathbf{ISR} & \underset{U_{KI}}{\overset{F_{IK}}{\rightleftarrows}} & \mathbf{Kleene} \\
 & \bot & U_I \big\downarrow \vdash \big\uparrow F_I & \bot & \\
 & & \mathbf{Set} & &
\end{array}
$$

$$
\begin{array}{ll}
F_B \stackrel{\text{def}}{=} F_{IB} \circ F_I & F_K \stackrel{\text{def}}{=} F_{IK} \circ F_I \\
U_B = U_I \circ U_{BI} & U_K = U_I \circ U_{KI}
\end{array}
$$

The units of $F_B \dashv U_B$ and $F_K \dashv U_K$ will be denoted by η' and η'', respectively.

Let us now explicitly state some of the facts constituting the adjunction $F_{IK} \dashv U_{KI}$. For each idempotent semiring $\mathbf{S}$ and Kleene algebra $\mathbf{K}$, the bijection from $\mathbf{Kleene}(F_{IK}(\mathbf{S}), \mathbf{K})$ to $\mathbf{ISR}(\mathbf{S}, U_{KI}(\mathbf{K}))$ is denoted by $\varphi_{\mathbf{S},\mathbf{K}}$. Indeed, for each arrow $f \colon \mathbf{S}' \to \mathbf{S}$ in $\mathbf{ISR}$ and $g \colon \mathbf{K} \to \mathbf{K}'$ in $\mathbf{ISR}$, the following diagram in $\mathbf{Set}$ commutes:

$$
\begin{array}{ccc}
\mathbf{Kleene}(F_{IK}(\mathbf{S}), \mathbf{K}) & \xrightarrow{\varphi_{\mathbf{S},\mathbf{K}}} & \mathbf{ISR}(\mathbf{S}, U_{KI}(\mathbf{K})) \\
{\scriptstyle \mathbf{Kleene}(F_{IK}(f), g)} \big\downarrow & & \big\downarrow {\scriptstyle \mathbf{ISR}(f, U_{KI}(g))} \\
\mathbf{Kleene}(F_{IK}(\mathbf{S}'), \mathbf{K}') & \xrightarrow{\varphi_{\mathbf{S}',\mathbf{K}'}} & \mathbf{ISR}(\mathbf{S}', U_{KI}(\mathbf{K}'))
\end{array}
$$

where $\mathbf{Kleene}(F_K(f), g)$ maps

$$\mathbf{Kleene}(F_{IK}(\mathbf{S}), \mathbf{K}) \ \ni \ h \mapsto g \circ h \circ F_{IK}(f) \ \in \ \mathbf{Kleene}(F_{IK}(\mathbf{S}'), \mathbf{K}')$$

and $\mathbf{ISR}(f, U_{KI}(f))$ does

$$\mathbf{ISR}(\mathbf{S}, U_{KI}(\mathbf{K})) \ \ni \ k \mapsto U_{KI}(g) \circ k \circ f \ \in \ \mathbf{ISR}(\mathbf{S}', U_{KI}(\mathbf{K}')) \ .$$

This property is called *naturality* of φ. The subscript $\mathbf{S}, \mathbf{K}$ will be omitted unless confusions occur. The next property follows from the naturality.

Lemma 2.5 *Diagram* (1) *commutes if and only if diagram* (2) *commutes.*

$$\begin{array}{ccc} F_{IK}(\mathbf{S}) & \xrightarrow{\;i\;} & \mathbf{K} \\ {\scriptstyle F_{IK}(f)}\big\downarrow & (1) & \big\downarrow{\scriptstyle g} \\ F_{IK}(\mathbf{S}') & \xrightarrow[\;j\;]{} & \mathbf{K}' \end{array} \qquad\qquad \begin{array}{ccc} \mathbf{S} & \xrightarrow{\;\varphi(i)\;} & U_{KI}(\mathbf{K}) \\ {\scriptstyle f}\big\downarrow & (2) & \big\downarrow{\scriptstyle U_{KI}(g)} \\ \mathbf{S}' & \xrightarrow[\;\varphi(j)\;]{} & U_{KI}(\mathbf{K}') \end{array}$$

Proof. Assume that (1) commutes. j is taken to $\varphi(j \circ F_{IK}(f))$ and $\varphi(j) \circ f$ by $\varphi \circ \mathbf{Kleene}(F_{IK}(f), \mathrm{id}_{\mathbf{K}})$ and $\mathbf{ISR}(f, U_{KI}(\mathrm{id}_{\mathbf{K}}) \circ \varphi$, respectively. By the naturality of φ, we have $\varphi(j \circ F_{IK}(f)) = \varphi(j) \circ f$. Similarly, considering $\mathbf{Kleene}(F_{IK}(\mathrm{id}_{\mathbf{S}}), g)$ and $\mathbf{ISR}(\mathrm{id}_{\mathbf{S}}, U_{KI}(g))$, we obtain $\varphi(g \circ i) = U_{KI}(g) \circ \varphi(i)$. By the assumption, we have $\varphi(j) \circ f = U_{KI}(g) \circ \varphi(i)$. The opposite direction is proved similarly using bijectivity in addition to naturality.

Proposition 2.6 *Let* $\mathbf{S} = (S, +, \cdot, 0, 1)$ *be an idempotent semiring such that, for each* $s \in S$*,* $s \leq 1$*. Then,* $F_{IK}(S) \cong (S, +, \cdot, {}^*, 0, 1)$*, where* * *maps each* $s \in S$ *to* 1*.*

Proof. Set $\mathbf{S}' = (S, +, \cdot, {}^*, 0, 1)$ and take $\varphi(\mathrm{id}_{F_{IK}(\mathbf{S})})\colon \mathbf{S} \to U_{KI}(F_{IK}(\mathbf{S}))$. Since, by the definition of $\mathbf{S}'$, $U_{KI}(\mathbf{S}') = \mathbf{S}$ and an idempotent semiring homomorphism $\varphi(\mathrm{id}_{F_{IK}(\mathbf{S})})$ is also a Kleene algebra homomorphism, the following diagram commutes.

$$\begin{array}{ccc} \mathbf{S} & \xrightarrow{\qquad \mathrm{id}_{\mathbf{S}} \qquad} & U_{KI}(\mathbf{S}')\ (= \mathbf{S}\,) \\ {\scriptstyle \mathrm{id}_{\mathbf{S}}}\big\downarrow & & \big\downarrow{\scriptstyle U_{KI}(\varphi(\mathrm{id}_{F_{IK}(\mathbf{S})}))\ (= \varphi(\mathrm{id}_{F_{IK}(\mathbf{S})})\,)} \\ \mathbf{S} & \xrightarrow[\quad \varphi(\mathrm{id}_{F_{IK}(\mathbf{S})}) \quad]{} & U_{KI}(F_{IK}(\mathbf{S})) \end{array}$$

By Lemma 2.5, the following diagram commutes.

$$\begin{array}{ccc} F_{IK}(\mathbf{S}) & \xrightarrow{\quad \varphi^{-1}(\mathrm{id}_{\mathbf{S}}) \quad} & \mathbf{S}' \\ {\scriptstyle F_{IK}(\mathrm{id}_{\mathbf{S}})}\big\downarrow & & \big\downarrow{\scriptstyle \varphi(\mathrm{id}_{F_{IK}(\mathbf{S})})} \\ F_{IK}(\mathbf{S}) & \xrightarrow[\quad \mathrm{id}_{F_{IK}(\mathbf{S})} \quad]{} & F_{IK}(\mathbf{S}) \end{array}$$

So, we have $\mathrm{id}_{F_{IK}(\mathbf{S})} = \varphi(\mathrm{id}_{F_{IK}(\mathbf{S})}) \circ \varphi^{-1}(\mathrm{id}_{\mathbf{S}})$. Since $\varphi(\mathrm{id}_{F_{IK}(\mathbf{S})})$ is a component of the unit of $F_{IK} \dashv U_{KI}$ with respect to $\mathbf{S}$, and $\mathbf{S} = U_{KI}(\mathbf{S}')$, we have $\mathrm{id}_{\mathbf{S}} = U_{KI}(\varphi^{-1}(\mathrm{id}_{\mathbf{S}})) \circ \varphi(\mathrm{id}_{F(\mathbf{S})})$. Here, $U_{KI}(\varphi^{-1}(\mathrm{id}_{\mathbf{S}}))$ is $\varphi^{-1}(\mathrm{id}_{\mathbf{S}})$ itself. Thus we have $\mathrm{id}_{\mathbf{S}} = \varphi^{-1}(\mathrm{id}_{\mathbf{S}}) \circ \varphi(\mathrm{id}_{F(\mathbf{S})})$.

3 Kleene algebras with tests

We provide a definition of Kleene algebras with tests. The definition is slightly more general than Kozen's one.

Definition 3.1 (Kleene Algebra with Tests) A Kleene algebra with tests is a triple $\langle \mathbf{B}, \mathbf{K}, i \rangle$ where

- $\mathbf{B}$ is a Boolean algebra,
- $\mathbf{K}$ is a Kleene algebra, and
- i is an idempotent semiring homomorphism from $U_{BI}(\mathbf{B}) \to U_{KI}(\mathbf{K})$.

The category of Kleene algebras with tests and their homomorphisms, which is denoted by **KAT**, is the comma category (U_{BI}, U_{KI}), that is, an arrow $\langle f, g \rangle$ from $\langle \mathbf{B}, \mathbf{K}, i \rangle$ to $\langle \mathbf{B}', \mathbf{K}', i' \rangle$ in **KAT** is a pair of a Boolean algebra homomorphism $f\colon \mathbf{B} \to \mathbf{B}'$ and a one of Kleene algebra $g\colon \mathbf{K} \to \mathbf{K}'$ such that the following diagram commutes.

$$\begin{array}{ccc} U_{BI}(\mathbf{B}) & \xrightarrow{i} & U_{KI}(\mathbf{K}) \\ {\scriptstyle U_{BI}(f)}\downarrow & & \downarrow{\scriptstyle U_{KI}(g)} \\ U_{BI}(\mathbf{B}') & \xrightarrow[i']{} & U_{KI}(\mathbf{K}') \end{array}$$

Kleene algebras with tests in the sense of Kozen [6] is a special case where i is an inclusion. For details of comma categories, see [8].

Definition 3.2 (free Kleene algebra with tests) A *free Kleene algebra with tests* generated by a pair (T, Σ) of sets T and Σ is defined to be a Kleene algebra with tests $\langle \mathbf{B}, \mathbf{K}, i \rangle$ and a pair (η_T, η_Σ) of maps $\eta_T\colon T \to B$ from T to the carrier set B of $\mathbf{B}$ and $\eta_\Sigma\colon \Sigma \to K$ from Σ to the carrier set K of $\mathbf{K}$ which satisfy the following universal property:

> for each Kleene algebra with tests $\langle \mathbf{B}', \mathbf{K}', i' \rangle$ and each pair (f, g) of maps $f\colon T \to B'$ from T to the carrier set B' of $\mathbf{B}'$ and $g\colon \Sigma \to K'$ from Σ to the carrier set K' of $\mathbf{K}'$ there is a unique arrow $\langle \hat{f}, \hat{g} \rangle\colon \langle \mathbf{B}, \mathbf{K}, i \rangle \to \langle \mathbf{B}', \mathbf{K}', i' \rangle$ in **KAT** such that
>
> $$f = \hat{f} \circ \eta_T \quad \text{and} \quad g = \hat{g} \circ \eta_\Sigma \ .$$

Kozen and Smith [7] showed existence of the free Kleene algebra with tests generated by a pair of finite sets through a construction of the set of languages of guarded strings. Since their result is based on Kozen's definition, i and i' in definition 3.2 are required to be inclusions. So it is not obvious whether the free algebra given by Kozen and Smith is the free algebra in our sense. After giving our construction of free Kleene algebra with tests (in Section 4), we compare Kozen-Smith's and ours (in Section 5).

We recall the definition of guarded strings and of regular sets of guarded strings.

Let T be a finite set. Then the Boolean algebra $F_B(T)$ is atomic. The set of atoms of $F_B(T)$ will be denoted by $\mathcal{A}_{F_B(T)}$. A guarded string over a finite set T and a (possibly finite) set Σ is a sequence $\alpha_0 p_0 \alpha_1 \cdots \alpha_{n-1} p_{n-1} \alpha_n$, where $n \geq 0$ and $\alpha_i \in \mathcal{A}_{F_B(T)}$ and $p_i \in \Sigma$. $\mathbf{Lang}_{T,\Sigma}$ denotes the powerset of the set of guarded strings over T and Σ. The fusion product $\diamond\colon \mathbf{Lang}_{T,\Sigma} \times \mathbf{Lang}_{T,\Sigma} \to \mathbf{Lang}_{T,\Sigma}$ is defined by

$$X \diamond Y \stackrel{\text{def}}{=} \{x\alpha y \mid x\alpha \in X, \alpha y \in Y\} \ .$$

Definition 3.3 ($\mathbf{Reg}_{T,\Sigma}$) The set $\mathbf{Reg}_{T,\Sigma}$ is defined inductively as follows:

- if $\alpha \in \mathcal{A}_{F_B(T)}$, then $\{\alpha\} \in \mathbf{Reg}_{T,\Sigma}$,
- if $p \in \Sigma$, then $\{\alpha p \beta \mid \alpha, \beta \in \mathcal{A}_{F_B(T)}\} \in \mathbf{Reg}_{T,\Sigma}$,
- if $X, Y \in \mathbf{Reg}_{T,\Sigma}$, then $X \cup Y \in \mathbf{Reg}_{T,\Sigma}$,
- if $X, Y \in \mathbf{Reg}_{T,\Sigma}$, then $X \diamond Y \in \mathbf{Reg}_{T,\Sigma}$,
- if $X \in \mathbf{Reg}_{T,\Sigma}$, then $\bigcup_{n \geq 0} X^n \in \mathbf{Reg}_{T,\Sigma}$, where $X^0 \stackrel{\text{def}}{=} \mathcal{A}_{F_B(T)}$ and $X^{n+1} \stackrel{\text{def}}{=} X \diamond X^n$.

Though Kozen and Smith require the finiteness of T and Σ, it is not necessary that Σ is finite. Elements of $\mathbf{Reg}_{T,\Sigma}$ are called regular sets. The empty set $\emptyset$ is a regular set since $\{\alpha\} \diamond \{\beta\} = \emptyset$ if $\alpha \neq \beta$. Since $\diamond$ over the power set $\wp(\mathcal{A}_{F_B(T)})$ of $\mathcal{A}_{F_B(T)}$ coincides with $\cap$,

$$\mathbf{B}_T = (\wp(\mathcal{A}_{F_B(T)}), \emptyset, \mathcal{A}_{F_B(T)}, \cup, \diamond, ^-)$$

is a Boolean algebra which is isomorphic to $F_B(T)$. Also

$$\mathbf{K}_{T,\Sigma} = (\mathbf{Reg}_{T,\Sigma}, \emptyset, \mathcal{A}_{F_B(T)}, \cup, \diamond, \bigcup_{n \geq 0} (-)^n)$$

is a Kleene algebra. Considering the inclusion $\subseteq$ from $\wp(\mathcal{A}_{F_B(T)})$ to $\mathbf{Reg}_{T,\Sigma}$, $\langle \mathbf{B}_T, \mathbf{K}_{T,\Sigma}, \subseteq \rangle$ is a Kleene algebra with tests.

In [3, 4] it was shown that the unital quantale $\mathbf{Q}_{T,\Sigma} = (\mathbf{Lang}_{T,\Sigma}, \mathcal{A}_{F_B(T)}, \diamond, \bigcup)$ can be presented to be a coproduct of unital quantales [9]:

> Unital quantales are monoids with arbitrary join in which left and right multiplications are universally additive. Left adjoints to the forgetful functors from the category of unital quantales to **Set** and to **ISR** exist, which are denoted by F_Q and F_{IQ}, respectively. Then $\mathbf{Q}_{T,\Sigma}$ is a coproduct of $F_{IQ}(U_{BI}(F_B(T)))$ and $F_{IQ}(\Sigma)$.

Analogously, $\mathbf{K}_{T,\Sigma}$ is presented to be a coproduct of Kleene algebras.

Define a homomorphism ι from $F_{IK}(U_{BI}(F_B(T)))$ to $\mathbf{K}_{T,\Sigma}$ to be a unique extension of an idempotent semiring homomorphism from $U_{BI}(F_B(T))$ to $U_{KI}(\mathbf{K}_{T,\Sigma})$ determined by $t \mapsto \{\alpha \in \mathcal{A}_{F_B(T)} \mid \alpha \le t\}$, that is,

$$(\varphi(\iota))(t) = \{\alpha \in \mathcal{A}_{F_B(T)} \mid \alpha \le t\} \ .$$

The idempotent semiring homomorphism $\varphi(\iota)$ will appear again in the proof of Theorem 5.1. Also define a homomorphism κ from $F_K(\Sigma)$ to $\mathbf{K}_{T,\Sigma}$ to be a unique extension of a mapping $\breve{\kappa}$ which maps

$$\Sigma \ \ni \ p \mapsto \{\alpha p \beta \mid \alpha, \beta \in \mathcal{A}_{F_B(T)}\} \ \in \ \mathbf{Reg}_{T,\Sigma}$$

induced from the universal property of the adjunction $F_K \dashv U_K$. Then the following holds.

Lemma 3.4 *The Kleene algebra* $\mathbf{K}_{T,\Sigma}$

$$F_{IK}(U_{BI}(F_B(T))) \xrightarrow{\ \iota\ } \mathbf{K}_{T,\Sigma} \xleftarrow{\ \kappa\ } F_K(\Sigma)$$

is a coproduct in **Kleene** *together with* ι *and* κ.

Proof. Given a Kleene algebra $\mathbf{K}$ and two Kleene algebra homomorphisms f from $F_{IK}(U_{BI}(F_B(T)))$ to $\mathbf{K}$ and g from $F_K(\Sigma)$ to $\mathbf{K}$, we define a Kleene algebra homomorphism h from $\mathbf{K}_{T,\Sigma}$ to $\mathbf{K}$ inductively:

- for $\alpha \in \mathcal{A}_{F_B(T)}$, $h(\{\alpha\}) = f'(\alpha)$, where f' is a mapping from the carrier set of $F_B(T)$ to the carrier set of $\mathbf{K}$ determined by $\varphi(f)$,
- for $p \in \Sigma$, $h(\{\alpha p \beta \mid \alpha, \beta \in \mathcal{A}_{F_B(T)}\}) = g_0(p)$ where g_0 is a map from Σ to $U_K(\mathbf{K})$ induced from the universal property of the adjunction $F_K \dashv U_K$,
- $h(X \cup Y) = h(X) + h(Y)$,
- $h(X \diamond Y) = h(X)h(Y)$,
- $h(\bigcup_{n \ge 0} X^n) = h(X)^*$,

By its definition, h preserves the structure of Kleene algebras except for the two nullary operators. Indeed, h preserves them, too, since $\emptyset$ is given by fusion product of two singleton sets of different atoms, $\mathcal{A}_{F_B(T)}$ is expressed by the union of the singleton sets of all atoms, and f' preserves $\wedge$ and $\vee$. For an element b of $U_BI(F_B(T))$, b may be expressed uniquely as the join $\alpha_1 + \cdots + \alpha_n$ of all atoms (which are only finitely many) such that $\alpha_i \le t$, and $U_{KI}(h)(\varphi(\iota)(b)) = f'(\alpha_1) + \cdots + f'(\alpha_n) = f'(\alpha_1) + \cdots + f'(\alpha_n) = f'(\alpha_1 + \cdots + \alpha_n) = \varphi(f)(b)$. Thus the diagram

$$\begin{array}{ccc}
U_{BI}(F_B(T)) & \xrightarrow{\ \varphi(\iota)\ } & U_{KI}(\mathbf{K}_{T,\Sigma}) \\
{\scriptstyle \mathrm{id}_{U_{BI}(F_B(T))}} \downarrow & & \downarrow {\scriptstyle U_{KI}(h)} \\
U_{BI}(F_B(T)) & \xrightarrow[\ \varphi(f)\ ]{} & U_{KI}(F_{IK}(U_{BI}(F_B(T))))
\end{array}$$

commutes. By Lemma 2.5, the diagram

$$\begin{array}{ccc}
F_{IK}(U_{BI}(F_B(T))) & \xrightarrow{\iota} & \mathbf{K}_{T,\Sigma} \\
\Big\downarrow{\scriptstyle F_{IK}(\mathrm{id}_{U_{BI}(F_B(T))})} & & \Big\downarrow{\scriptstyle h} \\
F_{IK}(U_{BI}(F_B(T))) & \xrightarrow[f]{} & F_{IK}(U_{BI}(F_B(T)))
\end{array}$$

commutes, and we have $f = h \circ \iota$. By the definition of κ, $\check{\kappa} = U_K(\kappa) \circ \eta''_\Sigma$. So, $U_K(h) \circ \check{\kappa} = U_K(h) \circ U_K(\kappa) \circ \eta_\Sigma = U_K(h \circ \kappa) \circ \eta_\Sigma$. Also, by the definition of h, $U_K(h)(\check{\kappa}(p)) = g_0(p)$. Moreover, by the definition of g_0, $g_0(p) = U_K(g) \circ \eta''_\Sigma$. Thus, by the uniqueness of g, we have $g = h \circ \kappa$. Assume $f = \overline{h} \circ \iota$ and $g = \overline{h} \circ \kappa$. Then, given $\alpha \in \mathcal{A}_{F_B(T)}$, $h(\{\alpha\}) = f'(\alpha) = \varphi(f)(\alpha)$ by the definition of h. Since $\varphi(f) = U_{KI}(f) \circ \varphi(\mathrm{id}_{F_{IK}(U_{BI}(F_B(T)))})$, and $U_{KI}(f)$ is f itself,

$$\varphi(f)(\alpha) = f(\varphi(\mathrm{id}_{F_{IK}(U_{BI}(F_B(T)))})(\alpha))$$

Using the assumption and the fact $\varphi(\iota) = \iota(\varphi(\mathrm{id}_{F_{IK}(U_{BI}(F_B(T)))}))$,

$$f(\varphi(\mathrm{id}_{F_{IK}(U_{BI}(F_B(T)))})(\alpha)) = \overline{h}(\iota(\varphi(\mathrm{id}_{F_{IK}(U_{BI}(F_B(T)))})(\alpha))) = \overline{h}(\{\alpha\})$$

So, $h(\{\alpha\}) = \overline{h}(\{\alpha\})$. Given $p \in \Sigma$

$$\begin{aligned}
&h(\{\alpha p \beta \mid \alpha, \beta \in \mathcal{A}_{F_B(T)}\}) = g_0(p) = \\
&\quad g(\eta''_\Sigma(p)) = \overline{h}(\kappa(\eta''_\Sigma(p))) = (\overline{h}(\{\alpha p \beta \mid \alpha, \beta \in \mathcal{A}_{F_B(T)}\})
\end{aligned}$$

Since both h and $\overline{h}$ are Kleene algebra homomorphisms, it holds that $h = \overline{h}$. Therefore h is a unique arrow in **Kleene** which satisfies $f = h \circ \iota$ and $g = h \circ \kappa$, which finishes the proof of the coproduct property.

4 Free construction of Kleene algebra with tests

In [3, 4], the existence of the free Kleene algebra with tests generated by a pair of (possibly infinite) sets was shown using the technique of finite limit sketches. With this technique, an explicit construction is not necessary for the proof, so none was given. This section provides an explicit construction using adjunctions and coproducts in **Kleene**.

If the forgetful functor $U\colon$ **KAT** $\to$ **Set**$\times$**Set** which takes an object $\langle \mathbf{B}, \mathbf{K}, f \rangle$ and an arrow $\langle h, k \rangle$ to a pair of their carrier sets (B, K) and a pair of morphisms (h, k) has a left adjoint, the image of the pair of sets (A, B) under the left adjoint together with the unit of the adjunction is the free Kleene algebra with tests generated by (A, B).

Since we already have the adjunctions $F_B \vdash U_B$ and $F_K \vdash U_K$, the functor $F_B \times F_K\colon$ **Set** $\times$ **Set** $\to$ **Bool** $\times$ **Kleene** is a left adjoint to $U_B \times U_K\colon$ **Bool** $\times$ **Kleene** $\to$ **Set** $\times$ **Set**.

Define Ψ to be the functor from **KAT** to **Bool** $\times$ **Kleene** which takes an object $\langle \mathbf{B}, \mathbf{K}, f \rangle$ and an arrow $\langle h, k \rangle$ to the pair $(\mathbf{B}, \mathbf{K})$ of algebras and the pair (h, k) of homomorphisms. Then it holds that $U = (U_B \times U_K) \circ \Psi$. So, if Ψ has a left adjoint, we obtain a left adjoint to U. Thus we may have the free Kleene algebra with tests generated by a pair of sets.

For a pair of a Boolean algebra $\mathbf{B}$ and a Kleene algebra $\mathbf{K}$, we have a Kleene algebra with tests

$$\langle \mathbf{B}, F_{IK}(U_{BI}(\mathbf{B})) + \mathbf{K}, \varphi(i) \rangle$$

where i is the first injection of coproduct

$$F_{KI}(U_{BI}(\mathbf{B})) \xrightarrow{i} F_{KI}(U_{BI}(\mathbf{B})) + \mathbf{K} \xleftarrow{j} \mathbf{K}$$

And for a pair of a Boolean algebra homomorphism $f\colon \mathbf{B} \to \mathbf{B}'$ and a Kleene algebra homomorphism $g\colon \mathbf{K} \to \mathbf{K}'$ we have two idempotent semiring homomorphisms $U_{BI}(f)$ and $U_{KI}(F_{IK}(U_{BI}(f)) + g)$. Then these two satisfy the following.

Proposition 4.1 *$\varphi(i') \circ U_{BI}(f) = U_{KI}(F_{IK}(U_{BI}(f)) + g) \circ \varphi(i)$ where i and i' are the first injections of the coproducts $F_{IK}(U_{BI}(\mathbf{B})) + \mathbf{K}$ and $F_{IK}(U_{BI}(\mathbf{B}')) + \mathbf{K}'$, respectively.*

Proof. By the definition of $F_{IK}(U_{BI}(f)) + g$ the diagram

$$\begin{array}{ccc}
F_{IK}(U_{BI}(\mathbf{B})) & \xrightarrow{i} & F_{IK}(U_{BI}(\mathbf{B})) + \mathbf{K} \\
{\scriptstyle F_{IK}(U_{BI}(f))}\downarrow & & \downarrow{\scriptstyle F_{IK}(U_{BI}(f)) + g} \\
F_{IK}(U_{BI}(\mathbf{B}')) & \xrightarrow[i']{} & F_{IK}(U_{BI}(\mathbf{B}')) + \mathbf{K}'
\end{array}$$

commutes. So the diagram

$$\begin{array}{ccc}
U_{BI}(\mathbf{B}) & \xrightarrow{\varphi(i)} & U_{KI}(F_{IK}(U_{BI}(\mathbf{B})) + \mathbf{K}) \\
{\scriptstyle U_{BI}(f)}\downarrow & & \downarrow{\scriptstyle U_{KI}(F_{IK}(U_{BI}(f)) + g)} \\
U_{BI}(\mathbf{B}') & \xrightarrow[\varphi(i')]{} & U_{KI}(F_{IK}(U_{BI}(\mathbf{B}')) + \mathbf{K}')
\end{array}$$

commutes by Lemma 2.5.

Therefore $\langle f, F_{IK}(U_{BI}(f)) + g \rangle$ is an arrow from $\langle \mathbf{B}, F_{IK}(U_{BI}(\mathbf{B})) + \mathbf{K}, \varphi(i) \rangle$ to $\langle \mathbf{B}, F_{IK}(U_{BI}(\mathbf{B}')) + \mathbf{K}', \varphi(i') \rangle$ in **KAT**.

Φ is the functor which takes an object $(\mathbf{B}, \mathbf{K})$ and an arrow (f, g) in **Bool** $\times$ **Kleene** to $\langle \mathbf{B}, \mathbf{K}, \varphi(i) \rangle$ and $\langle f, F_{IK}(U_{BI}(f)) + g \rangle$ in **KAT**, respectively.

Theorem 4.2 *Φ is a left adjoint to Ψ.*

Proof. Define the mapping $\xi_{(\mathbf{B},\mathbf{K}),\langle \mathbf{B}',\mathbf{K}',i'\rangle}\colon \mathbf{KAT}(\Phi(\mathbf{B},\mathbf{K}),\langle \mathbf{B}',\mathbf{K}',i'\rangle) \to \mathbf{Bool}\times\mathbf{Kleene}((\mathbf{B},\mathbf{K}),\Psi(\langle \mathbf{B}',\mathbf{K}',i'\rangle))$ for each object $(\mathbf{B},\mathbf{K})$ in $\mathbf{Bool}\times\mathbf{Kleene}$ and $\langle \mathbf{B}',\mathbf{K}',i'\rangle$ in $\mathbf{KAT}$ as follows:

$$\left\langle \begin{array}{c} \mathbf{B} \\ f\big\downarrow \\ \mathbf{B}' \end{array}, \begin{array}{c} F_{KI}(U_{BI}(\mathbf{B}))+\mathbf{K} \\ \big\downarrow g \\ \mathbf{K}' \end{array} \right\rangle \mapsto \left(\begin{array}{c} \mathbf{B} \\ f\big\downarrow \\ \mathbf{B}' \end{array}, \begin{array}{c} \mathbf{K} \\ \big\downarrow g\circ j \\ \mathbf{K}' \end{array} \right)$$

where j is the second injection of $F_{KI}(U_{BI}(\mathbf{B}))+\mathbf{K}$. In the sequel, ξ means $\xi_{(\mathbf{B},\mathbf{K}),\langle \mathbf{B}',\mathbf{K}',i'\rangle}$. It is sufficient to show that ξ is bijective. Assume that $\xi(\langle f,g\rangle) = \xi(\langle f',g'\rangle)$, that is, $f = f'$ and $g\circ j = g'\circ j$. Since $\langle f,g\rangle$ and $\langle f',g'\rangle$ are arrows in $\mathbf{KAT}$, the following diagram commutes both for $y = g$ and for $y = g'$.

$$\begin{array}{ccc} U_{BI}(\mathbf{B}) & \xrightarrow{\varphi(i)} & U_{KI}(F_{IK}(U_{BI}(\mathbf{B}))+\mathbf{K}) \\ U_{BI}(f)\big\downarrow & & \big\downarrow U_{KI}(y) \qquad (y = g \text{ or } g') \\ U_{BI}(\mathbf{B}') & \xrightarrow[i']{} & U_{KI}(\mathbf{K}') \end{array}$$

So, by Lemma 2.5, the diagram

$$\begin{array}{ccc} F_{IK}(U_{BI}(\mathbf{B})) & \xrightarrow{i} & F_{IK}(U_{BI}(\mathbf{B}))+\mathbf{K} \\ F_{IK}(U_{BI}(f))\big\downarrow & & \big\downarrow y \qquad (y = g \text{ or } g') \\ F_{IK}(U_{BI}(\mathbf{B}')) & \xrightarrow[\varphi^{-1}(i')]{} & \mathbf{K}' \end{array}$$

commutes again. Since $g\circ j = g'\circ j$ is assumed, $g = g'$ by the uniqueness of intermediating arrow of $F_{IK}(U_{BI}(\mathbf{B}))+\mathbf{K}$ with respect to $\varphi^{-1}(i')\circ F_{IK}(U_{BI}(f))$ and $g'\circ j$. Therefore, ξ is one-to-one. Given an arrow $(h,k)\colon (\mathbf{B},\mathbf{K}) \to \Psi(\langle \mathbf{B}',\mathbf{K}',i'\rangle)$ in $\mathbf{Bool}\times\mathbf{Kleene}$, we obtain two arrows $U_{BI}(h)$ and $U_{KI}(m)$ in $\mathbf{ISR}$, where m is a unique intermediating arrow of $F_{IK}(U_{BI}(\mathbf{B}))+\mathbf{K}$ with respect to $\varphi^{-1}(i')\circ F_{IK}(U_{BI}(h))$ and k. By the definition of m, the diagram

$$\begin{array}{ccc} F_{IK}(U_{BI}(\mathbf{B})) & \xrightarrow{i} & F_{IK}(U_{BI}(\mathbf{B}))+\mathbf{K} \\ F_{IK}(U_{BI}(h))\big\downarrow & & \big\downarrow m \\ F_{IK}(U_{BI}(\mathbf{B}')) & \xrightarrow[\varphi^{-1}(i')]{} & \mathbf{K}' \end{array}$$

commutes. So, by Lemma 2.5, the following diagram commutes, too.

$$\begin{array}{ccc}
U_{BI}(\mathbf{B}) & \xrightarrow{\varphi(i)} & U_{KI}(F_{IK}(U_{BI}(\mathbf{B})) + \mathbf{K}) \\
{\scriptstyle U_{BI}(h)}\downarrow & & \downarrow{\scriptstyle U_{KI}(m)} \\
U_{BI}(\mathbf{B}') & \xrightarrow[i']{} & U_{KI}(\mathbf{K}')
\end{array}$$

So, $\langle h, m\rangle$ is an arrow from $\Phi(\mathbf{B}, \mathbf{K})$ to $\langle \mathbf{B}', \mathbf{K}', i'\rangle$ in **KAT**. Moreover, since $\xi(\langle h, m\rangle) = (h, m \circ j)$ and $m \circ j = k$, $\xi(\langle h, m\rangle) = (h, k)$. Therefore, ξ is onto.

Corollary 4.3 *A component of the unit of $\Phi \dashv \Psi$ with respect to an object $(\mathbf{B}, \mathbf{K})$ in* **Bool** $\times$ **Kleene** *is $(\mathrm{id}_{\mathbf{B}}, j)$, where j is the second injection of the coproduct $F_{IK}(U_{BI}(F_B(T))) + F_K(\Sigma)$.*

Proof. It is immediate from $\xi(\mathrm{id}_{\Phi(\mathbf{B},\mathbf{K})}) = (\mathrm{id}_{\mathbf{B}}, j)$.

Corollary 4.4 *$\Phi \circ (F_B \times F_K)$ is a left adjoint to U.*

Corollary 4.5 *$\langle F_B(T), F_{IK}(U_{BI}(F_B(T))) + F_K(\Sigma), \varphi(i)\rangle$ together with a pair $(\eta'_T, U_K(j) \circ \eta''_\Sigma)$ of maps η'_T and $U_K(j) \circ \eta''_\Sigma$ is the free Kleene algebra with tests generated by the pair (T, Σ) of (possibly infinite) sets, where i and j are injections of the coproduct $F_{IK}(U_{BI}(F_B(T))) + F_K(\Sigma)$, η' and η'' are the units of $F_B \dashv U_B$ and $F_K \dashv U_K$.*

5 Comparison

In this section Kozen-Smith's free Kleene algebra with tests $\langle \mathbf{B}_T, \mathbf{K}_{T,\Sigma}, \subseteq\rangle$ (see after Definition 3.3) and ours $\langle F_B(T), F_{IK}(U_{BI}(F_B(T)))+F_K(\Sigma), \varphi(i)\rangle$ are compared. Since Kozen-Smith's construction works in the case that (T, Σ) is a pair of finite set T and (possibly infinite) set Σ, we assume that T is finite.

Theorem 5.1 $\langle \mathbf{B}_T, \mathbf{K}_{T,\Sigma}, \subseteq\rangle \cong \langle F_B(T), F_{IK}(U_{BI}(F_B(T))) + F_K(\Sigma), \varphi(i)\rangle$.

Proof. Define a Boolean algebra homomorphism h from $F_B(T)$ to $\mathbf{B}_T$ by

$$t \mapsto \{\alpha \in \mathcal{A}_{F_B(T)} \mid \alpha \leq t\} \ .$$

It is known that h is an isomorphism. Lemma 3.4 has shown that $\mathbf{K}_{T,\Sigma}$ is a coproduct of $F_{IK}(U_{BI}(F_B(T)))$ and $F_K(\Sigma)$. So, define the Kleene algebra homomorphism k from $F_{IK}(U_{BI}(F_B(T))) + F_K(\Sigma)$ to $\mathbf{K}_{T,\Sigma}$ to be the unique intermediating arrow with respect to injections ι and κ of $\mathbf{K}_{T,\Sigma}$, then k is an isomorphism. By the definition of k, the diagram

$$\begin{array}{ccc} F_{IK}(U_{BI}(F_B(T))) & \xrightarrow{i} & F_{IK}(U_{BI}(F_B(T))) + F_K(\Sigma) \\ {\scriptstyle F_{IK}(\mathrm{id}_{U_{BI}(F_B(T))})}\downarrow & & \downarrow{\scriptstyle k} \\ F_{IK}(U_{BI}(F_B(T))) & \xrightarrow[\iota]{} & \mathbf{K}_{T,\Sigma} \end{array}$$

commutes. So, by Lemma 2.5, the diagram

$$\begin{array}{ccc} U_{BI}(F_B(T)) & \xrightarrow{\varphi(i)} & U_{KI}(F_{IK}(U_{BI}(F_B(T))) + F_K(\Sigma)) \\ {\scriptstyle \mathrm{id}_{U_{BI}(F_B(T))}}\downarrow & & \downarrow{\scriptstyle U_{KI}(k)} \\ U_{BI}(F_B(T)) & \xrightarrow[\varphi(\iota)]{} & U_{KI}(\mathbf{K}_{T,\Sigma}) \end{array}$$

commutes. Moreover, by the definition of ι, the diagram

$$\begin{array}{ccc} U_{BI}(F_B(T)) & & \\ {\scriptstyle U_{BI}(h)}\downarrow & \searrow{\scriptstyle \varphi(\iota)} & \\ U_{BI}(\mathbf{B}_T) & \xrightarrow[\subseteq]{} & U_{KI}(K_{T,\Sigma}) \end{array}$$

commutes. Therefore $\langle h, k\rangle$ is an isomorphism in **KAT**.

Corollary 5.2 *$\langle \mathbf{B}_T, \mathbf{K}_{T,\Sigma}, \subseteq\rangle$ together with the pair of mappings*

$$\begin{array}{rcl} T \ni t & \mapsto & \{\alpha \in \mathcal{A}_{F_B(T)} \mid \alpha \leq \eta'_T(t)\} \in \wp(\mathcal{A}_{F_B(T)}) \quad \textit{and} \\ \Sigma \ni p & \mapsto & \{\alpha p \beta \mid \alpha, \beta \in \mathcal{A}_{F_B(T)}\} \in \mathbf{Reg}_{T,\Sigma} \end{array}$$

is the free Kleene algebra with tests in our sense, where η' is the unit of $F_B \dashv U_B$.

Since the third component of Kozen-Smith's free Kleene algebra with tests is inclusion, Theorem 5.1 gives rise that if T is finite, the third component of our free Kleene is one-to-one. In general, however, it is not yet clear whether the third component of our free Kleene algebra with tests is always one-to-one or not.

For a (possibly infinite) set T, the greatest element and the least element of $F_B(T)$ are not equal. So, by Proposition 2.6, $F_{IK}(U_{BI}(F_B(T)))$ is non-trivial. Also, by Remark 2.4, $F_K(\Sigma)$ is integral. Thus we have the following property.

Theorem 5.3 *The third component $\varphi(i)$ of the free Kleene algebra with tests $\langle F_B(T), F_{IK}(U_{BI}(F_B(T))) + F_K(\Sigma), \varphi(i)\rangle$ is one-to-one for each pair of sets (T, Σ).*

Proof. Take the free Kleene algebra $\langle F_B(T), F_{IK}(U_{BI}(F_B(T)))+F_K(\Sigma), \varphi(i)\rangle$. $U_{KI}(i)$ is one-to-one since, by Theorem 5.3, i is one-to-one and $U_{KI}(i)$ is i itself. Replace **S** with $U_{BI}(F_B(T))$ in the proof of Proposition 2.6. Then, $\varphi(i) = U_{KI}(i) \circ \varphi(\mathrm{id}_{F_{IK}(U_{BI}(F_B(T)))})$ since $\varphi(\mathrm{id}_{F_{IK}(U_{BI}(F_B(T)))})$ is a component of the unit of $F_{IK} \dashv U_{KI}$ with respect to $U_{BI}(F_B(T))$. Here, $\varphi(\mathrm{id}_{F_{IK}(U_{BI}(F_B(T)))})$ is not only a homomorphism of idempotent semirings but also a homomorphism of Kleene algebras from $U_{BI}(F_B(T))'$ to $F_{IK}(U_{BI}(F_B(T)))$ by the definition of $U_{BI}(F_B(T))'$. Moreover, by Proposition 2.6, $\varphi(\mathrm{id}_{F_{IK}(U_{BI}(F_B(T)))})$ is an isomorphism from $U_{BI}(F_B(T))'$ to $F_{IK}(U_{BI}(F_B(T)))$. Thus, $\varphi(\mathrm{id}_{F_{IK}(U_{BI}(F_B(T)))})$ is one-to-one. Therefore, $\varphi(i)$ is one-to-one.

Corollary 5.4 *Infinitely-generated free Kleene algebras with tests in the sense of Kozen exist.*

6 Conclusion

Starting from slightly more general definition of Kleene algebras with tests than Kozen's definition, we have provided a free construction of Kleene algebras with tests. Though the starting point is different from Kozen and Smith, the results of both constructions are isomorphic to each other. Since our construction has been given as a combination of basic notions such as adjunctions between **Kleene** and the categories of related algebraic structures, and coproducts in **Kleene**, the free algebras are generated quite systematically. Especially, the bijective correspondence φ provided by the adjunction $F_{IK} \dashv U_{KI}$ and the notion of coproduct in **Kleene** were important for our free construction. The notion of coproduct was the key notion in the comparison between our construction and Kozen and Smith's. The fact that the Kleene algebras consisting of regular sets of guarded strings are coproducts in **Kleene** may be helpful for more mathematical investigation of these Kleene algebras. Moreover this systematic manner allowed us to obtain the result of the existence of free Kleene algebras with tests generated by infinitely many generators without extra effort.

Acknowledgement: The author would like to thank Millie Rhoss de Guzman, Jules Desharnais, Wolfram Kahl, Ridha Khedri, and Michael Winter for discussions and helpful comments.

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