Right-linear finite path overlapping rewrite systems effectively preserve recognizability*

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Abstract

Right-linear finite path overlapping TRS are shown to effectively preserve recognizability. The class of right-linear finite path overlapping TRS properly includes the class of linear generalized semi-monadic TRS and the class of inverse left-linear growing TRS, which are known to effectively preserve recognizability.

1 Introduction

Much effort has been devoted to finding subclasses of TRSs which have reasonable computational power and for which important problems are decidable and, if possible, efficiently solvable. Tree automata inherit many favorable properties of finite state automata on strings[5]. For a tree automaton $\mathcal{A}$, let $\mathcal{L}(\mathcal{A})$ be the set of terms accepted by $\mathcal{A}$. A set T of terms is *recognizable* if there is a tree automaton $\mathcal{A}$ such that $T = \mathcal{L}(\mathcal{A})$. The class of recognizable sets is closed under boolean operations (union, intersection and complementation), and the emptiness problem is decidable for a recognizable set. If TRSs and recognizable sets of terms can be related appropriately, then these favorable properties of recognizable sets help us solve some problems in TRSs.

Two different directions for relating TRS and recognizable sets exist. One direction is the study of a TRS which effectively preserves recognizability[2, 6, 7, 8, 11, 13]. For a TRS $\mathcal{R}$ and a set T of terms, define $(\to^*_{\mathcal{R}})(T) = \{t \mid \exists s \in T, s \to^*_{\mathcal{R}} t\}$. A TRS $\mathcal{R}$ is said to *effectively preserve recognizability* if, for any tree automaton $\mathcal{A}$, $(\to^*_{\mathcal{R}})(\mathcal{L}(\mathcal{A}))$ is also recognizable and a tree automaton $\mathcal{A}_*$

*This paper is partially based on [16].

such that $(\rightarrow^*_{\mathcal{R}})(\mathcal{L}(\mathcal{A})) = \mathcal{L}(\mathcal{A}_*)$ can be effectively constructed. Joinability, reachability and local confluence are decidable for a TRS which effectively preserves recognizability[7, 8]. Since it is undecidable whether a given TRS effectively preserves recognizability or not[6], decidable subclasses of TRSs which effectively preserve recognizability have been investigated. Such classes include ground TRS[1], right-linear monadic TRS[13], linear semi-monadic TRS[2] and linear generalized semi-monadic TRS[8]. Another direction of the study for relating TRS and recognizable sets is to find a class of TRS R such that the set $(\leftarrow^*_{\mathcal{R}})(\mathcal{L}(\mathcal{A})) = \{t \mid \exists s \in \mathcal{L}(\mathcal{A}), t \rightarrow^*_{\mathcal{R}} s\}$ is recognizable for any tree automaton $\mathcal{A}$[4, 10, 12]. A linear growing TRS[10] has this property, and later, the result was extended to left-linear growing TRS[12]. Obviously, if a TRS $\mathcal{R}$ has this property, then $\mathcal{R}^{-1} = \{l \rightarrow r \mid r \rightarrow l \in \mathcal{R}\}$ preserves recognizability, and vice versa. A TRS $\mathcal{R}$ is (right-)linear semi-monadic if and only if $\mathcal{R}^{-1}$ is (left-)linear growing except that the variable restriction (l is not a variable and $\mathcal{V}ar(r) \subseteq \mathcal{V}ar(l)$ for each $l \rightarrow r \in R$) is dropped in the definition of growing TRS[10, 12]. (See Section 3 for more details.)

In this paper, a new class of TRSs, right-linear *finite path overlapping* TRS is proposed (Section 4). A TRS in the class effectively preserves recognizability (Section 5), and the class properly includes known decidable classes of TRSs which effectively preserve recognizability. Our results positively solve the conjecture in [7] that right-linear semi-monadic term rewriting systems effectively preserve recognizability.

2 Preliminaries

We use the usual notions for terms, substitutions, etc (see [3] for details). Let $\mathcal{F}$ be a *signature* and $\mathcal{V}$ be an enumerable set of *variables*. An element in $\mathcal{F}$ is called a *function symbol* and the *arity* of $f \in \mathcal{F}$ is denoted by $a(f)$. A function symbol c with $a(c) = 0$ is called a *constant*. The set of *terms*, defined in the usual way, is denoted by $\mathcal{T}(\mathcal{F}, \mathcal{V})$. The set of variables occurring in t is denoted by $\mathcal{V}ar(t)$. A term t is *ground* if $\mathcal{V}ar(t) = \emptyset$. The set of ground terms is denoted by $\mathcal{T}(\mathcal{F})$. A ground term in $\mathcal{T}(\mathcal{F})$ may also be called an $\mathcal{F}$*-term*. A term is *linear* if no variable occurs more than once in the term. A *context* is a term which has exactly one special constant $\square \notin \mathcal{F}$. A term obtained from a context C by replacing $\square$ with a term s is written as $C[s]$. A relation R on a set T of terms is *closed under contexts* if $s\ R\ t$ implies $C[s]\ R\ C[t]$ for any context C and for any terms $s, t \in T$. A *substitution* σ is a mapping from $\mathcal{V}$ to $\mathcal{T}(\mathcal{F}, \mathcal{V})$, and written as $\sigma = \{x_1 \mapsto t_1, \ldots, x_n \mapsto t_n\}$ where t_i with $1 \leq i \leq n$ is a term which substitutes for the variable x_i. The term obtained by applying a substitution σ to a term t is written as $t\sigma$. $t\sigma$ is called an *instance* of t and t is said to *subsume* $t\sigma$. A *position* in a term t is defined as a sequence of positive integers as usual, and the set of all positions in a term t is denoted by $\mathcal{P}os(t)$. Let λ be the empty sequence which denotes the root position. If a position o_1 is a prefix (resp. proper prefix) of o_2, then we write $o_1 \preceq o_2$ (resp. $o_1 \prec o_2$). Two positions o_1 and o_2 are *disjoint* if neither $o_1 \preceq o_2$ nor $o_2 \preceq o_1$. A subterm of t at a position o is denoted by t/o. t/o is said to occur at *depth* $|o|$. The *depth* of a term t is $\max\{|o| \mid o \in \mathcal{P}os(t)\}$. If a term t is obtained from a term t' by replacing the subterms of t' at positions $o_1, \ldots, o_m$ ($o_i \in \mathcal{P}os(t'), o_i$ and o_j are disjoint if $i \neq j$) with terms $t_1, \ldots, t_m$, respectively, then we write $t = t'[o_i \leftarrow t_i \mid 1 \leq i \leq m]$.

A *rewrite rule* is an ordered pair of terms, written as $l \to r$. The variable restriction ($\mathcal{V}ar(r) \subseteq \mathcal{V}ar(l)$ and l is not a variable) is not assumed in this paper unless stated otherwise. A *term rewriting system* (*TRS*) is a finite set of rewrite rules. For a TRS $\mathcal{R}$, let $\mathcal{R}^{-1} = \{r \to l \mid l \to r \in \mathcal{R}\}$. For terms t, t' and a TRS $\mathcal{R}$, we write $t \to_{\mathcal{R}} t'$ if there exist a position $o \in \mathcal{P}os(t)$, a substitution σ and a rewrite rule $l \to r \in \mathcal{R}$ such that $t/o = l\sigma$ and $t' = t[o \leftarrow r\sigma]$. Define $\to^*_{\mathcal{R}}$ (resp. $\leftrightarrow^*_{\mathcal{R}}$) to be the reflexive and transitive (resp. the reflexive, symmetric and transitive) closure of $\to_{\mathcal{R}}$. Also the positive closure of $\to_{\mathcal{R}}$ is denoted by $\to^+_{\mathcal{R}}$. The subscript $\mathcal{R}$ of $\to_{\mathcal{R}}$ is omitted if $\mathcal{R}$ is clear from the context. A *redex* (*in* $\mathcal{R}$) is an instance of l for some $l \to r \in \mathcal{R}$. A *normal form* (in $\mathcal{R}$) is a term which has no redex as its subterm. Let $\mathrm{NF}_{\mathcal{R}}$ denote the set of all ground normal forms in $\mathcal{R}$. A rewrite rule $l \to r$ is *left-linear*(resp. *right-linear*) if l is linear (resp. r is linear). A rewrite rule is *linear* if it is left-linear and right-linear. A TRS $\mathcal{R}$ is *left-linear* (resp. *right-linear, linear*) if every rule in $\mathcal{R}$ is left-linear (resp. right-linear, linear).

Notions such as *reachability*, *joinability*, *unifiability*, *unifier*, *most general unifier*, *confluence* and *local confluence* are defined in the usual way. For a TRS $\mathcal{R}$, two terms t_1 and t_2 are *$\mathcal{R}$-unifiable* if there exists a substitution σ such that $t_1\sigma \leftrightarrow^*_{\mathcal{R}} t_2\sigma$.

A *tree automaton*(*TA*)[5] is defined by a 4-tuple $\mathcal{A} = (\mathcal{F}, \mathcal{Q}, \mathcal{Q}_{\mathit{final}}, \Delta)$ where $\mathcal{F}$ is a signature, $\mathcal{Q}$ is a finite set of states, $\mathcal{Q}_{\mathit{final}} \subseteq \mathcal{Q}$ is a set of final states, and Δ is a finite set of transition rules of the form $f(q_1, \ldots, q_n) \to q$ where $f \in \mathcal{F}$, $a(f) = n$, and $q_1, \ldots, q_n, q \in \mathcal{Q}$ or of the form $q' \to q$ where $q, q' \in \mathcal{Q}$. A rule with the former form is called a *non-ε-rule* and a rule with the latter form is called an *ε-rule*. Consider the set of ground terms $\mathcal{T}(\mathcal{F} \cup \mathcal{Q})$ where we define $a(q) = 0$ for $q \in \mathcal{Q}$. A *move* of a TA can be regarded as a rewrite relation on $\mathcal{T}(\mathcal{F} \cup \mathcal{Q})$ by regarding transition rules in Δ as rewrite rules on $\mathcal{T}(\mathcal{F} \cup \mathcal{Q})$. For terms t and t' in $\mathcal{T}(\mathcal{F} \cup \mathcal{Q})$, we write $t \vdash_{\mathcal{A}} t'$ if and only if $t \to_{\Delta} t'$. The reflexive and transitive closure of $\vdash_{\mathcal{A}}$ is denoted by $\vdash^*_{\mathcal{A}}$. For a TA $\mathcal{A}$ and $t \in \mathcal{T}(\mathcal{F})$, if $t \vdash^*_{\mathcal{A}} q_f$ for a final state $q_f \in \mathcal{Q}_{\mathit{final}}$, then we say t is *accepted* by $\mathcal{A}$. The set of ground terms accepted by $\mathcal{A}$ is denoted by $\mathcal{L}(\mathcal{A})$. A set T of ground terms is *recognizable* if there is a TA $\mathcal{A}$ such that $T = \mathcal{L}(\mathcal{A})$. Also let $\mathcal{L}_q(\mathcal{A}) = \{t \mid t \vdash^*_{\mathcal{A}} q\}$ for a state q of $\mathcal{A}$. Recognizable sets inherit some useful properties of regular (string) languages[5].

Lemma 2.1 *The class of recognizable sets is effectively closed under union, intersection and complementation. For a recognizable set T, the following problems are decidable. (1) Does a given ground term t belong to T? (2) Is T empty?* □

3 TRS which Preserves Recognizability

For a TRS $\mathcal{R}$ and a set T of ground terms, define $(\rightarrow^*_{\mathcal{R}})(T) = \{t \mid \exists s \in T$ such that $s \rightarrow^*_{\mathcal{R}} t\}$ and $(\leftarrow^*_{\mathcal{R}})(T) = \{t \mid \exists s \in T$ such that $t \rightarrow^*_{\mathcal{R}} s\}$. A TRS $\mathcal{R}$ is said to *effectively preserve recognizability* if, for any TA $\mathcal{A}$, the set $(\rightarrow^*_{\mathcal{R}})(\mathcal{L}(\mathcal{A}))$ is also recognizable and we can effectively construct a TA which accepts $(\rightarrow^*_{\mathcal{R}})(\mathcal{L}(\mathcal{A}))$. In this paper, the class of TRSs which effectively preserve recognizability is written as EPR-TRS.

Theorem 3.1 *If a TRS $\mathcal{R}$ belongs to EPR-TRS, then the reachability relation and the joinability relation for $\mathcal{R}$ are decidable[7]. It is also decidable whether $\mathcal{R}$ is locally confluent or not[8].* □

Theorem 3.2 *Let $\mathcal{R}$ be a TRS such that $\mathcal{R}^{-1}$ is an EPR-TRS and $\mathcal{R}$ is left-linear, then it is decidable whether $\mathcal{R}$ is weakly normalizing or not.*

Proof. It is easily understood that $\mathcal{T}(\mathcal{F}) = (\leftarrow^*_{\mathcal{R}})(NF_{\mathcal{R}})$ if and only if $\mathcal{R}$ is weakly normalizing. On the other hand, the set $NF_{\mathcal{R}}$ of normal forms for $\mathcal{R}$ is recognizable since $\mathcal{R}$ is left-linear. Since $(\leftarrow^*_{\mathcal{R}})(T) = (\rightarrow^*_{\mathcal{R}^{-1}})(T)$ for any set T of terms and $\mathcal{R}^{-1}$ is in EPR-TRS, we can see that $\overline{(\leftarrow^*_{\mathcal{R}})(NF_{\mathcal{R}})}$ is recognizable. Note that $\mathcal{T}(\mathcal{F}) = (\leftarrow^*_{\mathcal{R}})(NF_{\mathcal{R}})$ if and only if $\overline{(\leftarrow^*_{\mathcal{R}})(NF_{\mathcal{R}})} = \emptyset$ ($\emptyset$ denotes the empty set). Hence, $\mathcal{T}(\mathcal{F}) = (\leftarrow^*_{\mathcal{R}})(NF_{\mathcal{R}})$ is decidable by Lemma 2.1. □

Theorem 3.3 *For a confluent $\mathcal{R} \in$ EPR-TRS and linear terms t_1 and t_2 with $\mathcal{V}ar(t_1) \cap \mathcal{V}ar(t_2) = \emptyset$, it is decidable whether t_1 and t_2 are $\mathcal{R}$-unifiable or not.*

Proof. Since $\mathcal{R}$ is confluent, t_1 and t_2 are $\mathcal{R}$-unifiable if and only if there exists a ground substitution σ and a term v such that $t_1\sigma \rightarrow^*_{\mathcal{R}} v$ and $t_2\sigma \rightarrow^*_{\mathcal{R}} v$. For a term t, let $I(t)$ denote the set of ground instances of t, i.e., $I(t) =$

$\{t\sigma \mid \sigma$ is a ground substitution$\}$. Then t_1 and t_2 are $\mathcal{R}$-unifiable if and only if

$$(\to_{\mathcal{R}}^*)(I(t_1)) \cap (\to_{\mathcal{R}}^*)(I(t_2)) \neq \emptyset \tag{3.1}$$

since $\mathcal{V}ar(t_1) \cap \mathcal{V}ar(t_2) = \emptyset$. It is easy to see that $I(t)$ is recognizable for any linear term t. Thus $(\to_{\mathcal{R}}^*)(I(t_1))$ and $(\to_{\mathcal{R}}^*)(I(t_2))$ are recognizable since $\mathcal{R} \in$ EPR-TRS. By Lemma 2.1, the condition (3.1) is decidable. □

Unfortunately it is undecidable whether a given TRS belongs to EPR-TRS or not[6]. Therefore decidable subclasses of EPR-TRS have been proposed. Among them are ground TRS (G-TRS) by Brainerd[1], right-linear monadic TRS (RL-M-TRS) by Salomaa[13], linear semi-monadic TRS (L-SM-TRS) by Coquidé et al.[2], and linear generalized semi-monadic TRS (L-GSM-TRS) by Gyenizse and Vágvölgyi[8]. In [7], Gilleron and Tison conjectured that the class of right-linear semi-monadic TRSs is also included in EPR-TRS. Note that these papers assume the variable restriction.

Theorem 3.4 *G-TRS $\subset$ RL-M-TRS $\subset$ EPR-TRS, and G-TRS $\subset$ L-SM-TRS $\subset$ L-GSM-TRS $\subset$ EPR-TRS.* □

There is another stream of studies which relate TRSs and recognizability[10, 4, 12]. A TRS $\mathcal{R}$ (without the variable restriction) is *growing* if all variables in $\mathcal{V}ar(l) \cap \mathcal{V}ar(r)$ occur at depth 0 or 1 in l for every rewrite rule $l \to r$ in $\mathcal{R}$[10]. Nagaya and Toyama[12] showed that for each left-linear growing TRS (LL-G-TRS) $\mathcal{R}$, $\mathcal{R}^{-1}$ effectively preserves recognizability. If a TRS $\mathcal{R}$ satisfies the variable restriction then $\mathcal{R}$ is (linear, right-linear) semi-monadic if and only if $\mathcal{R}^{-1}$ is (linear, left-linear) growing and the left-hand side of every rewrite rule in $\mathcal{R}$ is not a constant. LL-G-TRS^{-1} properly includes both of RL-M-TRS and L-SM-TRS, and it is incomparable with L-GSM-TRS.

4 FPO-TRSs

A new class of TRS named *finite path overlapping TRS* (*FPO-TRS*) is proposed in this section without assuming the variable restriction. As we will show later, the class of RL-FPO-TRS properly includes the class of RL-GSM-TRS and LL-G-TRS^{-1}. It will also be shown in the next section that an RL-FPO-TRS (without the variable restriction) is an EPR-TRS. To the authors' knowledge, the proposed class is the largest decidable subclass of EPR-TRS. To define the class, some additional definitions are necessary. We say that a term s *sticks out of* t if t is not a variable and there is a variable position γ $(\neq \lambda)$ of t such that

1. for any position o with $\lambda \preceq o \prec \gamma$, we have $o \in \mathcal{P}os(s)$ and the function symbol of s at o and the function symbol of t at o are the same, and

2. $\gamma \in \mathcal{P}os(s)$ and s/γ is not a ground term.

When the position γ is of interest, we say that s sticks out of t at γ. If s sticks out of t at γ and s/γ is not a variable (i.e. s/γ is a non-ground and non-variable term), then s is said to *properly stick out of* t (at γ). For example, a term $f(g(x), a)$ sticks out of $f(g(y), b)$ at the position $1 \cdot 1$, and $f(g(g(x)), a)$ properly sticks out of $f(g(y), b)$ at the position $1 \cdot 1$. A *finite path overlapping term rewriting system* (*FPO-TRS*) is a TRS $\mathcal{R}$ such that the following *sticking-out graph* of $\mathcal{R}$ does not have a cycle of weight one or more.

Definition 4.1 The *sticking-out graph* of a TRS $\mathcal{R}$ is a directed graph $G = (V, E)$ where $V = \mathcal{R}$ (i.e. the vertices are the rewrite rules in $\mathcal{R}$) and E is defined as follows. Let v_1 and v_2 be (possibly identical) vertices which correspond to rewrite rules $l_1 \to r_1$ and $l_2 \to r_2$, respectively. Replace each variable in $\mathcal{V}ar(r_i) \setminus \mathcal{V}ar(l_i)$ with a fresh constant, say $\diamond$, for $i = 1, 2$.

1. If r_2 properly sticks out of a subterm of l_1, then E contains an edge from v_2 to v_1 with weight one.

2. If a subterm of r_2 properly sticks out of l_1, then E contains an edge from v_2 to v_1 with weight one.

3. If a subterm of l_1 sticks out of r_2, then E contains an edge from v_2 to v_1 with weight zero.

4. If l_1 sticks out of a subterm of r_2, then E contains an edge from v_2 to v_1 with weight zero.

□

An RL-TRS (right-linear TRS) being FPO is written as RL-FPO-TRS. The four cases are illustrated in Fig. 1.

Example 4.1 Let $\mathcal{R}_1 = \{\ p_1\colon f(x, a) \to f(h(y), x),\ p_2\colon g(y) \to f(g(y), b)\ \}$. Fig. 2 shows the sticking-out graph of $\mathcal{R}_1$. The right-hand side of p_2 properly sticks out of the left-hand side of p_1 at the position 1, and hence there is an edge of weight one from p_2 to p_1. The sticking-out graph also has a self-looping edge of weight zero at p_2 since the left-hand side $g(y)$ of p_2 sticks out of $f(g(y), b)/1 = g(y)$. Since the variable y in p_1 is replaced with a constant $\diamond$, the right-hand side of p_1 does not stick out of its left-hand side. There is

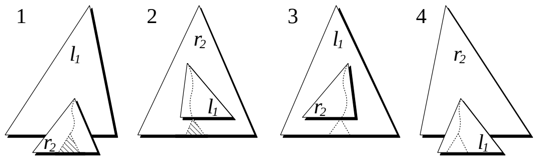

Figure 1: The sticking-out relations of rewrite rules.

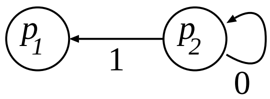

Figure 2: The sticking-out graph of $\mathcal{R}_1$.

no other edge since there is no other sticking-out relation between subterms of these rewrite rules. The sticking-out graph has a cycle of weight zero, but does not have a cycle of weight one or more, and hence $\mathcal{R}$ is finite path overlapping.

Let $\mathcal{R}_2 = \{f(x) \to g(f(g(x)))\}$. The subterm $f(g(x))$ of the right-hand side of the (unique) rewrite rule properly sticks out of its left-hand side, as in Condition 2 of the definition of sticking-out graph. The sticking-out graph of $\mathcal{R}_2$ consists of one vertex and one cycle with weight one. Therefore, $\mathcal{R}_2$ is not finite path overlapping. Note that $\mathcal{R}_2 \notin$ EPR-TRS since $(\to^*_{\mathcal{R}_2})(\{f(a)\}) = \{g^n(f(g^n(a))) \mid n \geq 0\}$ is not recognizable. □

Remark that the sticking-out graph is effectively constructible for a given TRS $\mathcal{R}$, and hence it is decidable whether a given TRS $\mathcal{R}$ is finite path overlapping or not (in $O(m^2n^2)$ time where m is the maximum size of a term in $\mathcal{R}$ and n is the number of rules in $\mathcal{R}$).

Although a generalized semi-monadic TRS (GSM-TRS) was originally defined with the variable restriction in [8], we define GSM-TRS without the variable restriction to treat growing TRS, GSM-TRS and FPO-TRS in a uniform way.

A TRS $\mathcal{R}$ is *generalized semi-monadic* if the following condition holds for any pair of (possibly the same) rewrite rules $l_1 \to r_1$ and $l_2 \to r_2$ in $\mathcal{R}$[8]. For $i = 1, 2$, each variable in $\mathcal{V}ar(r_i) \setminus \mathcal{V}ar(l_i)$ is replaced with a fresh constant. For any positions $\alpha \in \mathcal{P}os(l_1)$ and $\beta \in \mathcal{P}os(r_2)$ such that $\alpha = \lambda$ or $\beta = \lambda$ and for any term l_3 which subsumes l_1/α, if r_2/β and l_3 are unifiable, then

1. l_1/α is a variable, or

2. for any $\gamma \in \mathcal{P}os(l_3)$ such that $l_1/\alpha \cdot \gamma$ is a variable, $(l_3/\gamma)\sigma$ is a variable or a ground term where σ is the most general unifier of r_2/β and l_3.

Lemma 4.1 *A TRS $\mathcal{R}$ is in GSM-TRS if and only if the sticking-out graph of $\mathcal{R}$ has no edge with weight one. If a TRS $\mathcal{R}$ is generalized semi-monadic, then $\mathcal{R}$ is finite path overlapping.*

Proof. We show the only if part by contradiction. If part can be shown in a similar way. Assume that $\mathcal{R}$ is a GSM-TRS and contains rules $l_1 \to r_1$ and $l_2 \to r_2$ (each variable in $\mathcal{V}ar(r_i)\backslash\mathcal{V}ar(l_i)$ has been replaced with a constant $\diamond$ for $i = 1, 2$) which satisfy condition 1 of the definition of sticking-out graph. In this case, there is a position $\alpha \in \mathcal{P}os(l_1)$ such that r_2 properly sticks out of l_1/α. Let γ be the variable position of l_1/α at which r_2 properly sticks out of l_1/α, then $l_1/\alpha \cdot \gamma$ is a variable and r_2/γ is a non-ground and non-variable term. Let l_3 be the term which satisfies the following conditions. (1) For a position o with $\lambda \preceq o \prec \gamma$, l_3 and l_1/α have the same symbol at o, (2) a variable, say x_o, occurs at a position o which is disjoint to γ and is written as $o' \cdot i$ with $o' \prec \gamma$ and (3) a variable x_γ occurs at γ. It is easily understood that l_3 subsumes l_1/α and that l_3 and r_2 are unifiable by an mgu σ which in particular replaces x_γ by r_2/γ. Now we have $(l_3/\gamma)\sigma = r_2/\gamma$, which is neither a variable nor a ground term by assumption. This concludes that $\mathcal{R}$ is not a GSM-TRS. In a similar way, we can show that if any pair of rules in $\mathcal{R}$ satisfy the condition 2 of the definition of sticking-out graph, then $\mathcal{R}$ is not a GSM-TRS. $\square$

Theorem 4.2 *The class of RL-FPO-TRS properly includes the class of RL-GSM-TRS.*

Proof. The class of RL-FPO-TRS includes the class of RL-GSM-TRS by Lemma 4.1. TRS $\mathcal{R}_1$ in Example 4.1 is RL-FPO but not GSM. If we take $l_1 = f(x, a)$, $r_2 = f(g(y), b)$, $\alpha = \beta = \lambda$ and $l_3 = f(x, z)$, then r_2 and l_3 are unifiable by an mgu $\sigma = \{x \mapsto g(y), z \mapsto b\}$. Let $\gamma = 1$, then $l_1/\alpha \cdot \gamma = l_1/1$ is a variable x while $(l_3/\gamma)\sigma = g(y)$ is neither a variable nor a ground term. Therefore $\mathcal{R}_1$ is not a GSM-TRS. $\square$

5 Construction of tree automata

In this section, we will show that every RL-FPO-TRS $\mathcal{R}$ belongs to EPR-TRS by constructing a TA $\mathcal{A}_*$ such that $\mathcal{L}(\mathcal{A}_*) = (\to^*_{\mathcal{R}})(\mathcal{L}(\mathcal{A}))$ for a given TA $\mathcal{A}$.

To deal with non-left-linear TRS, we need to construct a kind of product automata whose states are Cartesian products of sets of terms. To represent such a Cartesian product and a usual first-order term in a uniform way, we introduce a packed state. Intuitively, a packed state is an extension of a first-order term such that a finite set of terms, rather than a single term, occurs at a subterm position. For a signature $\mathcal{F}$ and a finite set $\mathcal{Q}$, the set of *packed states*, denoted $\mathcal{P}_{\mathcal{F},\mathcal{Q}}$, is defined as follows:

1. If $q \in \mathcal{Q}$, then $\{q\} \in \mathcal{P}_{\mathcal{F},\mathcal{Q}}$.
2. If $f \in \mathcal{F}$ and $p_1, \ldots, p_{a(f)} \in \mathcal{P}_{\mathcal{F},\mathcal{Q}}$, then $\{f(p_1, \ldots, p_{a(f)})\} \in \mathcal{P}_{\mathcal{F},\mathcal{Q}}$.
3. If $p_1, p_2 \in \mathcal{P}_{\mathcal{F},\mathcal{Q}}$, then $p_1 \cup p_2 \in \mathcal{P}_{\mathcal{F},\mathcal{Q}}$.

For the readability, a packed state $\{t_1, \ldots, t_n\}$ is written as $\langle t_1, \ldots, t_n \rangle$. For example, let $\mathcal{F} = \{f, g\}$ with $a(f) = 2$ and $a(g) = 1$ and $\mathcal{Q} = \{q_1, q_2\}$. For example, we can easily verify that $\langle f(\langle q_1 \rangle, \langle q_2 \rangle), g(\langle g(\langle q_1 \rangle), \langle q_2 \rangle \rangle)\rangle$ belongs to $\mathcal{P}_{\mathcal{F},\mathcal{Q}}$.

Procedure 5.1 (Tree automata Construction)

Input: a TA $\mathcal{A} = (\mathcal{F}, \mathcal{Q}, \mathcal{Q}_{\mathit{final}}, \Delta)$ and an RL-TRS $\mathcal{R}$

Output: a TA $\mathcal{A}_*$ such that $\mathcal{L}(\mathcal{A}_*) = (\to_R^*)(\mathcal{L}(\mathcal{A}))$

Step 1. Add a new state q_{any} to $\mathcal{Q}$ and add a transition rule $f(q_{any}, \ldots, q_{any}) \to q_{any}$ to Δ for each f in $\mathcal{F}$. Obviously, $t \vdash_{\mathcal{A}}^* q_{any}$ for any $t \in \mathcal{T}(\mathcal{F})$. Let $\mathcal{A}_0 = (\mathcal{F}, \mathcal{Q}_0, \mathcal{Q}^0_{\mathit{final}}, \Delta_0)$ be a "packed" version of $\mathcal{A}$ where $\mathcal{Q}_0 = \{\langle q \rangle \mid q \in \mathcal{Q}\} \subseteq \mathcal{P}_{\mathcal{F},\mathcal{Q}}$, $\mathcal{Q}^0_{\mathit{final}} = \{\langle q \rangle \mid q \in \mathcal{Q}_{\mathit{final}}\}$, and $\Delta_0 = \{f(\langle q_1 \rangle, \ldots, \langle q_n \rangle) \to \langle q \rangle \mid f(q_1, \ldots, q_n) \to q \in \Delta\} \cup \{\langle q' \rangle \to \langle q \rangle \mid q' \to q \in \Delta\}$.

Step 2. Let $k = 0$. This k is used as a loop counter.

Step 3. Let $\mathcal{Q}_{k+1} = \mathcal{Q}_k$ and $\Delta_{k+1} = \Delta_k$.

Step 4. The set of transition rules is modified in this step. Let $l \to r$ be a rewrite rule in $\mathcal{R}$. Assume l has m variables $x_1, \ldots, x_m$ and x_i $(1 \le i \le m)$ occurs for γ_i times at positions o_{ij} $(1 \le j \le \gamma_i)$ in l. Also assume x_i occurs at o_i in r for $x_i \in \mathcal{V}ar(r)$. If there are states $p_{ij}, p \in \mathcal{Q}_k$ with $1 \le i \le m, 1 \le j \le \gamma_i$,

$$l[o_{ij} \leftarrow p_{ij} \mid 1 \le i \le m, 1 \le j \le \gamma_i] \vdash_k^* p \tag{5.1}$$

and

$$\mathcal{L}_{p_{i1}}(\mathcal{A}_k) \cap \cdots \cap \mathcal{L}_{p_{i\gamma_i}}(\mathcal{A}_k) \neq \emptyset \tag{5.2}$$

for $1 \leq i \leq m$, then add

$$p_i = \bigcup_{1 \leq j \leq \gamma_i} p_{ij} \quad (1 \leq i \leq m) \tag{5.3}$$

to $\mathcal{Q}_{k+1}$ as new states and let $\rho = \{x_i \mapsto p_i \mid 1 \leq i \leq m\} \cup \{x \mapsto \langle q_{any} \rangle \mid x \in \mathcal{V}ar(r) \setminus \mathcal{V}ar(l)\}$. If r is a variable, then let $t_{r\rho} = r\rho$. Otherwise, let $t_{r\rho} = \langle r\rho \rangle$. Do the following (a) through (c).

(a) Add $t_{r\rho} \to p$ to Δ_{k+1}.

(b) Let $p = \langle t_1, \ldots, t_n \rangle$. Add $t_{r\rho} \to \langle t_i \rangle$ to Δ_{k+1} for $1 \leq i \leq n$. A transition rule defined in (a) or (b) is called a *rewriting transition rule* of *degree* $k+1$ and if a move of the TA is caused by such a rule, then the move is called a *proper rewriting move* of *degree* $k+1$.

(c) Execute **ADDTRANS**$(t_{r\rho})$. In **ADDTRANS**$(t_{r\rho})$, new states and transition rules are defined so that $r\rho \vdash_{k+1}^{*} t_{r\rho}$.

Simultaneously execute this Step 4 for every rewrite rule and every tuple of states that satisfy conditions (5.1) and (5.2).

Step 5. Continue the loop until $\Delta_{k+1} = \Delta_k$. If $\Delta_{k+1} \neq \Delta_k$, then $k = k+1$ and go to Step 3.

Step 6. Output $\mathcal{A}_k$ as $\mathcal{A}_*$. □

Procedure 5.2 [ADDTRANS] This procedure takes a packed state p as an input. If p has already been defined as a state, then the procedure performs nothing. Otherwise, the procedure first defines p as a new state of $\mathcal{Q}_{k+1}$ and also defines transition rules as follows. It is required that if $p = \langle t_1, \ldots, t_n \rangle$ $(n \geq 2)$, then each $\langle t_i \rangle$ has been defined as a state.

Case 1. If $p = \langle c \rangle$ with c a constant, then define $c \to \langle c \rangle$ as a transition rule.

Case 2. If $p = \langle f(p_1, \ldots, p_{a(f)}) \rangle$ with $f \in \mathcal{F}$, then define $f(p'_1, \ldots, p'_{a(f)}) \to p$ as a transition rule where $p'_i = p_i$ if p_i is a state, otherwise $p'_i = \langle p_i \rangle$ for $1 \leq i \leq a(f)$ and execute **ADDTRANS**(p'_i) for $1 \leq i \leq a(f)$.

Case 3. If $p = \langle t_1, \ldots, t_n \rangle$ $(n \geq 2)$, then do the following (i) through (iii).

(i) Define new ε-rules $p \to \langle t_i \rangle$ for $1 \leq i \leq n$.

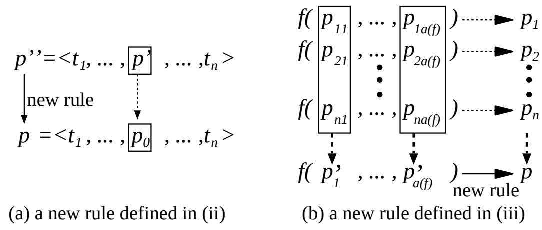

Figure 3: The new rules introduced by **ADDTRANS**.

(ii) For each transition rule of the form $p' \to p_0$ ($p', p_0 \in \mathcal{Q}_k, p_0 \subseteq p$), define a new ε-rule $p'' \to p$ and execute **ADDTRANS**(p'') where p'' is the state defined as $p'' = (p \setminus p_0) \cup p'$ (see Fig. 3(a)). In this case, if $p' \to p_0$ is a rewriting transition rule of degree k', then we call the new rule a *non-proper rewriting transition rule of degree* k'. If a move of the TA is caused by this new rule, then the move is also called a *non-proper rewriting move of degree* k'.

(iii) If there are states $p_1, \ldots, p_n$ and a function symbol f such that $p = \bigcup_{1 \le i \le n} p_i$ and $f(p_{i1}, \ldots, p_{ia(f)}) \to p_i \in \Delta_k$ for $1 \le i \le n$, then define new rules $f(p'_1, \ldots, p'_{a(f)}) \to p$ and $f(p'_1, \ldots, p'_{a(f)}) \to \langle t_i \rangle$ for $1 \le i \le n$ and execute **ADDTRANS**(p'_j) where $p'_j = \bigcup_{1 \le i \le n} p_{ij}$ for $1 \le j \le a(f)$ (see Fig. 3(b)). □

Example 5.1 Let $\mathcal{A} = (\mathcal{F}, \mathcal{Q}, \mathcal{Q}_{\mathit{final}}, \Delta)$ be a TA where $\mathcal{F} = \{f, g, h, c\}$ with $a(f) = 2$, $a(g) = a(h) = 1$ and $a(c) = 0$, $\mathcal{Q} = \{q_0, q_1, q'_0, q'_1, q'_2, q_f\}$, $\mathcal{Q}_{\mathit{final}} = \{q_f\}$ and Δ consists of the following transition rules:

$$\begin{array}{llll} c \to q_0, & h(q_0) \to q_1, & h(q_1) \to q_0, & \\ c \to q'_0, & h(q'_0) \to q'_1, & h(q'_1) \to q'_2, & h(q'_2) \to q'_0, \\ f(q_0, q'_0) \to q_f. & & & \end{array}$$

It can be easily verified that $\mathcal{L}(\mathcal{A}) = \{f(h^{2m}(c), h^{3n}(c)) \mid m, n \ge 0\}$. Let $\mathcal{R} = \{f(x, x) \to g(x), g(x) \to x\}$. $\mathcal{R}$ is an RL-FPO-TRS. We apply Procedure 5.1 to $\mathcal{A}$ and $\mathcal{R}$. Consider the rewrite rule $f(x, x) \to g(x)$ in Step 4 for $\mathcal{A}_0 (k = 0)$. Since a move $f(\langle q_0 \rangle, \langle q'_0 \rangle) \vdash_0 \langle q_f \rangle$ is possible, new transition rules

$$\underline{\langle g(\langle q_0, q'_0 \rangle) \rangle} \to \langle q_f \rangle \tag{5.4}$$

$$g(\underline{\langle q_0, q'_0 \rangle}) \to \langle g(\langle q_0, q'_0 \rangle) \rangle \tag{5.5}$$

$$c \to \langle q_0, q'_0 \rangle \tag{5.6}$$

$$h(\underline{\langle q_1, q'_2 \rangle}) \to \langle q_0, q'_0 \rangle$$

$$
\begin{array}{rcl}
h(\underline{\langle q_0, q_1' \rangle}) & \rightarrow & \langle q_1, q_2' \rangle \\
h(\underline{\langle q_1, q_0' \rangle}) & \rightarrow & \langle q_0, q_1' \rangle \\
h(\underline{\langle q_0, q_2' \rangle}) & \rightarrow & \langle q_1, q_0' \rangle \\
h(\underline{\langle q_1, q_1' \rangle}) & \rightarrow & \langle q_0, q_2' \rangle \\
h(\underline{\langle q_0, q_0' \rangle}) & \rightarrow & \langle q_1, q_1' \rangle
\end{array}
$$

are added to Δ_1 where **ADDTRANS** is recursively executed for the underlined subterms. The transition rule (5.4) is defined in Step 4 and (5.5) is added in Case 2 of **ADDTRANS**. When **ADDTRANS**($\langle q_0, q_0' \rangle$) is executed, the Case 3(iii) is applied to the input and the rule (5.6) is added by using the rules $c \rightarrow q_0$ and $c \rightarrow q_0'$. The others are also added in Case 3(iii) of **ADDTRANS**($\langle q_0, q_0' \rangle$) and in its recursive execution. Next, consider the rewrite rule $g(x) \rightarrow x$ in Step 4 for $\mathcal{A}_1$ ($k = 1$). Since

$$g(\langle q_0, q_0' \rangle) \vdash_1 \langle g(\langle q_0, q_0' \rangle) \rangle \vdash_1 \langle q_f \rangle,$$

$\langle q_0, q_0' \rangle \rightarrow \langle q_f \rangle$ is added to Δ_2. Thus we obtain

$$h(h(h(h(h(h(c)))))) \vdash_2^* \langle q_0, q_0' \rangle \vdash_2 \langle q_f \rangle$$

and hence $h(h(h(h(h(h(c)))))) \in \mathcal{L}(\mathcal{A}_2)$. We can verify that $\mathcal{A}_3 = \mathcal{A}_2$ $(= \mathcal{A}_*)$ and $\mathcal{L}(\mathcal{A}_*) = (\rightarrow_{\mathcal{R}}^*)(\mathcal{L}(\mathcal{A})) = \{g(h^{6n}(c)) \mid n \geq 0\} \cup \{h^{6n}(c) \mid n \geq 0\} \cup \mathcal{L}(\mathcal{A})$. □

5.1 Soundness

Lemma 5.1 *For any $k \geq 0$ and a state $q \in \mathcal{Q}_{final}$, $\mathcal{L}_q(\mathcal{A}_k) \subseteq (\rightarrow_{\mathcal{R}}^*)(\mathcal{L}_q(\mathcal{A}_0))$.* □

In order to prove Lemma 5.1, we need a kind of translation between TRSs which translates a non-left-linear TRS $\mathcal{R}$ to a left-linear conditional TRS which simulates $\mathcal{R}$ in the sense stated in Lemma 7.1. This translation is called linearization. We will postpone the definition and properties of linearization and the proof of Lemma 5.1 to Section 7.

5.2 Completeness

First we prove two technical lemmas concerning packed states.

Lemma 5.2 *For a positive integer n and states p_i, $\langle t_i \rangle$ $(1 \leq i \leq n)$ in $\mathcal{Q}_k$, if there is a state $\langle t_1, \ldots, t_n \rangle$ in $\mathcal{Q}_k$ and $p_i \vdash_k^* \langle t_i \rangle$ for $1 \leq i \leq n$, then $p \vdash_k^* \langle t_1, \ldots, t_n \rangle$ where $p = \bigcup_{1 \leq i \leq n} p_i$.*

Proof. If $n = 1$, then the lemma holds obviously. Consider the case $n \geq 2$. Assume that for each $1 \leq i \leq n$, $p_i = p_{i0} \vdash_k p_{i1} \vdash_k \cdots \vdash_k p_{il_i} = \langle t_i \rangle$ for some $l_i \geq 0$ and $\langle t_1, \ldots, t_n \rangle \in \mathcal{Q}_k$. If $l_i = 0$ for every $1 \leq i \leq n$, the lemma holds obviously. Assume that $l_i \geq 1$ for a particular i. Then $p_{il_{i-1}} \to \langle t_i \rangle \in \Delta_k$. Since $\langle t_1, \ldots, t_n \rangle \in \mathcal{Q}_k$, **ADDTRANS**$(\langle t_1, \ldots, t_n \rangle)$ has been executed in Procedure 5.1 and a new ε-rule $p' \to \langle t_1, \ldots, t_n \rangle$ is defined in Case 3(ii) where $p' = (\langle t_1, \ldots, t_n \rangle \setminus \langle t_i \rangle) \cup p_{il_{i-1}} = \langle t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_n \rangle \cup p_{il_{i-1}}$. Hence, the move $\langle t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_n \rangle \cup p_{il_{i-1}} \vdash \langle t_1, \ldots, t_n \rangle$ is possible and **ADDTRANS**$(\langle t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_n \rangle \cup p_{il_{i-1}})$ is recursively executed. Repeating the above argument, we have $p (= \bigcup_{1 \leq i \leq n} p_i) \vdash^* \langle t_1, \ldots, t_n \rangle$. □

Lemma 5.3 *For an $\mathcal{F}$-term s, and states $\langle t_i \rangle \in \mathcal{Q}_k$ with $1 \leq i \leq n$, if $s \vdash_k^* \langle t_i \rangle$ for $1 \leq i \leq n$ and $\langle t_1, \ldots, t_n \rangle \in \mathcal{Q}_k$, then $s \vdash_k^* \langle t_1, \ldots, t_n \rangle$.*

Proof. The lemma is shown by induction on the depth of the term s. If $s = c$ with $a(c) = 0$, then the sequence $s \vdash_k^* \langle t_i \rangle$ can be written as

$$c \vdash_k p_i \vdash_k^* \langle t_i \rangle \quad (1 \leq i \leq n) \tag{5.7}$$

for some $p_i \in \mathcal{Q}_k$. Since $p_i \vdash_k^* \langle t_i \rangle$ for $1 \leq i \leq n$, we obtain $p \vdash_k^* \langle t_1, \ldots, t_n \rangle$ where $p = \bigcup_{1 \leq i \leq n} p_i$ by Lemma 5.2. Since $c \to p_i \in \Delta_k$ and $p \in \mathcal{Q}_k$, the transition rule $c \to p$ is defined by Case 3(iii) of **ADDTRANS**(p). Therefore $c \vdash_k p \vdash_k^* \langle t_1, \ldots, t_n \rangle$.

Assume that the lemma holds for every term with depth $l - 1$ or less, and consider a term $s = f(s_1, \ldots, s_{a(f)})$ with depth l. The sequence $s \vdash_k^* \langle t_i \rangle$ can be written as

$$s \vdash_k^* f(p_{i1}, \ldots, p_{ia(f)}) \vdash_k p_i \vdash_k^* \langle t_i \rangle \qquad (1 \leq i \leq n) \tag{5.8}$$

where p_{ij} $(1 \leq j \leq a(f))$ and p_i are states. This implies $s_j \vdash_k^* p_{ij}$ for $1 \leq i \leq n$ and $1 \leq j \leq a(f)$, and therefore $s_j \vdash_k^* \bigcup_{1 \leq i \leq n} p_{ij}$ for $1 \leq j \leq a(f)$ by the induction hypothesis. Hence, the sequence

$$s \vdash_k^* f(\bigcup_{1 \leq i \leq n} p_{i1}, \ldots, \bigcup_{1 \leq i \leq n} p_{ia(f)}) \tag{5.9}$$

is possible. On the other hand, since all the moves in the sequence $p_i \vdash_k^* \langle t_i \rangle$ for $1 \leq i \leq n$ of (5.8) are ε-moves, transition rules are defined so that the sequence

$$p \vdash_k^* \langle t_1, \ldots, t_n \rangle \tag{5.10}$$

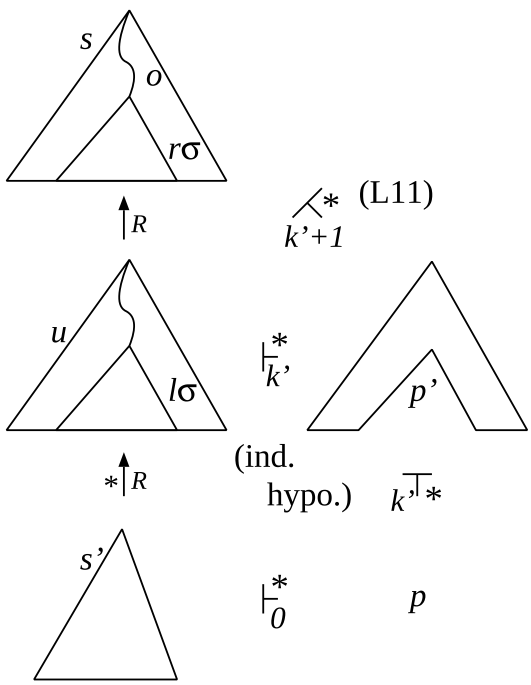

Figure 4: Sketch of the proof of Lemma 5.4.

where $p = \bigcup_{1 \leq i \leq n} p_i$ is possible by means of Lemma 5.2. Furthermore, by the move $f(p_{i1}, \ldots, p_{ia(f)}) \vdash_k p_i$ with $1 \leq i \leq n$ of (5.8) and by Case 3(iii) of **ADDTRANS**(p), the transition rule

$$f(\bigcup_{1 \leq i \leq n} p_{i1}, \ldots, \bigcup_{1 \leq i \leq n} p_{ia(f)}) \\ \to p \tag{5.11}$$

is defined. Summarizing (5.9),(5.10) and (5.11), we obtain $s \vdash_k^* \langle t_1, \ldots, t_n \rangle$. □

Lemma 5.4 *For a term $s \in (\to_{\mathcal{R}}^*)(\mathcal{L}(\mathcal{A}))$, there is an integer k such that $s \in \mathcal{L}(\mathcal{A}_k)$.*

Proof. It suffices to show that for a state $p \in \mathcal{Q}_0$, if $s' \to_{\mathcal{R}}^* s$ and $s' \in \mathcal{L}_p(\mathcal{A}_0)$, then there is an integer k such that $s \in \mathcal{L}_p(\mathcal{A}_k)$, or equivalently, $s \vdash_k^* p$. The claim is shown by induction on the length of the derivation $s' \to_{\mathcal{R}}^* s$. For the basis $s' = s$, the claim holds obviously. If $s' \to_{\mathcal{R}}^+ s$, then there is a term u such that $s' \to_{\mathcal{R}}^* u \to_{\mathcal{R}} s$. By induction hypothesis applied to $s' \to_{\mathcal{R}}^* u$, we have an integer k' such that $u \vdash_{k'}^* p$. Moreover, since $u \to_{\mathcal{R}} s$ there is a rewrite rule $l \to r \in R$, a substitution σ, and a position $o \in \mathcal{P}os(u)$ such that $u/o = l\sigma$ and $s = u[o \leftarrow r\sigma]$. Hence, there is a state $p' \in \mathcal{Q}_{k'}$ such that $u = u[o \leftarrow l\sigma] \vdash_{k'}^* u[o \leftarrow p'] \vdash_{k'}^* p$ and we have

$$l\sigma \vdash_{k'}^* p'. \tag{5.12}$$

Now, let us show that $r\sigma \vdash_{k'+1}^* p'$. Assume that l has m variables $x_1, \ldots, x_m$ and the variable x_i has γ_i occurrences in l at $o_{ij} \in \mathcal{P}os(l)$. By (5.12) there are states p_{ij} for $1 \leq i \leq m$ and $1 \leq j \leq \gamma_i$ such that

$$x_i\sigma \vdash_{k'}^* p_{ij} \tag{5.13}$$

and

$$l[o_{ij} \leftarrow p_{ij} \mid 1 \leq i \leq m, 1 \leq j \leq \gamma_i] \vdash_{k'}^* p'. \tag{5.14}$$

The sequence (5.13) means that $x_i\sigma \in \mathcal{L}_{p_{i1}}(\mathcal{A}_{k'}) \cap \cdots \cap \mathcal{L}_{p_{i\gamma_i}}(\mathcal{A}_{k'})$ and we have

$$\mathcal{L}_{p_{i1}}(\mathcal{A}_{k'}) \cap \cdots \cap \mathcal{L}_{p_{i\gamma_i}}(\mathcal{A}_{k'}) \neq \emptyset \tag{5.15}$$

for $1 \leq i \leq m$. By (5.14) and (5.15), a substitution $\rho = \{x_i \mapsto p_i \mid 1 \leq i \leq m\} \cup \{x \mapsto \langle q_{any} \rangle \mid x \in \mathcal{V}ar(r) \setminus \mathcal{V}ar(l)\}$ is defined in Step 4 of Procedure 5.1. By Lemma 5.3, each p_i in the co-domain of ρ satisfies

$$\mathcal{L}_{p_{i1}}(\mathcal{A}_{k'+1}) \cap \cdots \cap \mathcal{L}_{p_{i\gamma_i}}(\mathcal{A}_{k'+1}) \subseteq \mathcal{L}_{p_i}(\mathcal{A}_{k'+1}) \tag{5.16}$$

for $1 \leq i \leq m$ and transition rules are defined by **ADDTRANS** to satisfy that

$$r\rho \vdash_{k'+1}^* p'. \tag{5.17}$$

By (5.13) and (5.16), we have

$$x_i\sigma \vdash_{k'+1}^* p_i \qquad (1 \leq i \leq m). \tag{5.18}$$

Summarizing (5.17) and (5.18), we have $r\sigma \vdash_{k'+1}^* p'$, and the lemma holds since $s = u[o \leftarrow r\sigma] \vdash_{k'+1}^* u[o \leftarrow p'] \vdash_{k'}^* p$. □

By Lemma 5.1 and Lemma 5.4, we obtain the following theorem, which states the partial correctness of Procedure 5.1.

Theorem 5.5 *For an RL-TRS $\mathcal{R}$, if Procedure 5.1 halts then $\mathcal{L}(\mathcal{A}_*) = (\to_{\mathcal{R}}^*)(\mathcal{L}(\mathcal{A}))$.* □

5.3 Termination of Procedure 5.1

We show that if an RL-FPO-TRS is given to Procedure 5.1, then there is an upper-bound limit on the number of states which are newly defined. Once the set of states saturates, then the set of transition rules also saturates and the procedure halts. First, as a measure of the size of a state, we introduce the concept of the *layer* of a packed state. Intuitively, the number of layers of a packed state is the number of right-hand sides of rewrite rules which are used for defining the state. For a packed state $p \in \mathcal{Q}_k$, define the number of *layers* of p, denoted $\mathrm{layer}(p)$, as follows:

(1) if $p \in \mathcal{Q}_0$ or $p = \langle t \rangle$ with t a ground subterm of a rewrite rule in $\mathcal{R}$, then $\mathrm{layer}(p) = 0$,

(2) if $p = p_1 \cup p_2$, then $\mathrm{layer}(p) = \max\{\mathrm{layer}(p_1), \mathrm{layer}(p_2)\}$, and

(3) if $p = \langle r\sigma/o \rangle$ with $l \to r \in \mathcal{R}$, $o \in \mathcal{P}os(r)$, r/o is not a variable, $\mathcal{V}ar(r/o) = \{x_1, \ldots, x_n\}$ and $\sigma = \{x_i \mapsto p_i \mid 1 \leq i \leq n\}$, then $\mathrm{layer}(p) = 1 + \max\{\mathrm{layer}(p_i) \mid 1 \leq i \leq n\}$.

Remark that $\mathrm{layer}(p)$ is not defined for all packed states, but all packed states introduced in Procedure 5.1 are of the form (1), (2) or (3). Also remark that $\mathrm{layer}(p)$ is not always uniquely determined by this definition. If different values are defined as $\mathrm{layer}(p)$, then we choose the minimum among the values as $\mathrm{layer}(p)$. We note that in (3) above if $x_i \in \mathcal{V}ar(r) \setminus \mathcal{V}ar(l)$, then $p_i = \langle q_{any} \rangle$ and $\mathrm{layer}(p_i) = 0$. This means that variables which occurs only in the right-hand side are ignored for defining the number of layers.

Example 5.2 Consider the states of the TAs in Example 5.1. Let $l \to r = f(x, x) \to g(x) \in \mathcal{R}$, $o = \lambda$ and $\sigma = \{x \mapsto \langle q_0, q_0' \rangle\}$ in the above definition (3). Then, $p = \langle r\sigma/o \rangle = \langle g(\langle q_0, q_0' \rangle) \rangle$ and $\mathrm{layer}(p) = \mathrm{layer}(\langle q_0, q_0' \rangle) + 1 = \max\{\mathrm{layer}(\langle q_0 \rangle), \mathrm{layer}(\langle q_0' \rangle)\} + 1 = 1$. □

Lemma 5.6 *For any non-negative integer j, the number of packed states which have j or less layers is finite.*

Proof. The lemma will be shown by induction on j. For the base case, the number of the states that have 0 layer is finite, since the number of the states of $\mathcal{Q}_0$ and the number of the states that are made from ground subterms of the right-hand sides of a given TRS are finite.

Assume that the number of states that have $n - 1$ or less layers is finite and show it is also true for the case that $j = n$. In Procedure 5.1, there are four cases when a new state which has n layers is added.

1. In Step 4 of Procedure 5.1, a state which is defined as $p_i = \bigcup_{1\le j\le\gamma_i} p_{ij}$ in (5.3) is added.

2. In Step 4(c) of Procedure 5.1, a new state $t_{r\rho}$ is added.

3. In Case 2 of Procedure 5.2, a new state p'_i is added.

4. In Case 3(ii) of Procedure 5.2, a new state $p'' = (p' \setminus p_0) \cup p'$ is added.

5. In Case 3(iii) of Procedure 5.2, a new state $p'_j = \bigcup_{1\le i\le n} p_{ij}$ is added.

From the inductive hypothesis and the definition (3) of the number of layers, there exists a number k' such that case 2 does not take place at any loop counter k'' for $k'' \ge k'$ in Procedure 5.1. Let $\widetilde{\mathcal{Q}}_{k'} = \{t \mid t \in p, p \in \mathcal{Q}_{k'}\}$. (Note that a packed state itself is a set.) A new state which is added in case 1, 3, 4, or 5 is a subset of $\widetilde{\mathcal{Q}}_{k'}$. Since $\mathcal{Q}_{k'}$ is finite, the number of subsets of $\widetilde{\mathcal{Q}}_{k'}$ is also finite. Hence the lemma holds. □

In the following, it is shown that if $\mathcal{R}$ is an RL-FPO-TRS, then layer(p) $\le |\mathcal{R}|$ for any state p defined by Procedure 5.1 where $|\mathcal{R}|$ is the number of rewrite rules in $\mathcal{R}$. An outline of the proof is as follows. First we associate each rule in $\mathcal{R}$ with a non-negative integer called a *rank*. If $\mathcal{R}$ is finite path overlapping, then the rank is well-defined and is less than $|\mathcal{R}|$. Next, it is shown that if a rewrite rule with rank j is used in Step 4 of Procedure 5.1, then layer(p) $\le j+1$ for any state p defined in the same step. The *rank* of a rule in $\mathcal{R}$ is defined based on the sticking-out graph $G = (V, E)$ of $\mathcal{R}$. Let v be the vertex of G which corresponds to a rewrite rule $l \to r$ in $\mathcal{R}$. The *rank* of $l \to r$ is the maximum weight of a path to v from any vertex in V. If $\mathcal{R}$ is finite path overlapping, then the rank of any rewrite rule is a non-negative integer less than $|\mathcal{R}|$. For $\mathcal{R}_1$ in Example 4.1, the ranks of p_1 and p_2 are one and zero, respectively, since there is an edge with weight one from p_2 to p_1.

Lemma 5.7 *Let $l \to r$ be a rewrite rule and $\rho = \{x_i \mapsto p_i \mid 1 \le i \le m\} \cup \{x \mapsto \langle q_{any}\rangle \mid x \in \mathcal{V}ar(r) \setminus \mathcal{V}ar(l)\}$ be a substitution which are used in Step 4 of Procedure 5.1. If the rank of $l \to r$ is j or less, then layer(p_i) $\le j$ for each $1 \le i \le m$.* □

Before presenting a proof of the lemma, we first see how the number of layers of the state changes by a move of the TA. A transition rule of the TA is either an ε-rule or a non-ε-rule. An ε-rule is either an ε-rule of the original TA $\mathcal{A}_0$ or a rule defined in Step 4(a) or (b) of Procedure 5.1, or a rule defined in Case 3(i) or (ii) of **ADDTRANS** procedure. If an ε-rule of the original automaton is used at a move, then the number of layer does not change at the move. A non-ε-rule is either a non-ε-rule of $\mathcal{A}_0$, or a rule defined in Cases 1, 2 or 3(iii) of **ADDTRANS**. In all cases, the maximum number of layers in a state is increased by one or not changed by a move (Lemma 5.8). Hence, if the number of layers decreases at a move, then the rule is an ε-rule defined in Step 4(a) or (b) of Procedure 5.1 or in Case 3(ii) of **ADDTRANS**.

Lemma 5.8 *For a non-ε rule $f(p_1, \ldots, p_{a(f)}) \to p \in \Delta_k$ ($a(f) \geq 1$), let $m = \max\{\text{layer}(p_j) \mid 1 \leq j \leq a(f)\}$. Then, $m \leq \text{layer}(p) \leq m+1$.*

Proof. By induction on k. A non-ε rule is introduced either Step 1 of Procedure 5.1, or Case 1, Case 2, or Case 3(iii) of **ADDTRANS**. If $f(p_1, \ldots, p_{a(f)}) \to p$ is introduced in Step 1, then $\max\{\text{layer}(p_i) \mid 1 \leq i \leq a(f)\} = 0$ and $\text{layer}(p) = 0$. Thus the lemma holds. If $c \to \langle c \rangle$ is introduced in Case 1 of **ADDTRANS**, then the lemma holds vacuously. Assume that $f(p_1, \ldots, p_{a(f)}) \to p = \langle f(p_1, \ldots, p_{a(f)}) \rangle$ is introduced in Case 2. Then there exists a rewrite rule $l \to r$ and a $\mathcal{Q}_k$-substitution ρ which satisfies (5.1) and (5.2) such that $(r/o)\rho = f(p_1, \ldots, p_{a(f)})$ for some $o \in \mathcal{P}os(r)$. Let $m = \max\{\text{layer}(p_j) \mid 1 \leq j \leq a(f)\}$. By definition of layer($\cdot$), $\text{layer}(p) = m$. Assume that $f(p_1, \ldots, p_{a(f)}) \to p$ is introduced in Case 3(iii). Let

$$\begin{aligned} m &= \max\{\text{layer}(p_j) \mid 1 \leq j \leq a(f)\} \\ &= \max\{\text{layer}(p_{ij}) \mid 1 \leq i \leq n, 1 \leq j \leq a(f)\}. \end{aligned}$$

There are two cases for each $1 \leq i \leq n$. If $\text{layer}(p_{ij}) = m$ for some j ($1 \leq j \leq a(f)$), then $m \leq \text{layer}(\langle t_i \rangle) \leq m+1$ by the inductive hypothesis. If $\text{layer}(p_{ij}) < m$ for each j ($1 \leq j \leq a(f)$), then $\text{layer}(\langle t_i \rangle) \leq m$ by the inductive hypothesis. Hence, $m \leq \text{layer}(p) = \max\{\text{layer}(\langle t_i \rangle) \mid 1 \leq i \leq n\} \leq m+1$. □

Proof of Lemma 5.7 The proof is by induction on the loop variable k of Procedure 5.1. When $k = 0$, every state belongs to $\mathcal{Q}_0$ and $\text{layer}(p_i) = 0$ for $1 \leq i \leq n$, and the lemma holds for any j. Assume that the lemma holds for $k \leq n-1$, and consider the case with $k = n$. The inductive part is shown by contradiction. Without loss of generality, let p_1 be a state such that $\text{layer}(p_1) \geq j+1$. Since $p_1 = \bigcup_{1 \leq l \leq \gamma_1} p_{1l}$, $\text{layer}(p_1) = \max\{\text{layer}(p_{1l}) \mid 1 \leq l \leq \gamma_1\}$ by the definition of layer($\cdot$). We can assume p_{11} is the state such that $\text{layer}(p_{11}) \geq j+1$ without loss of generality. Let us consider the sequence (5.1) in Step 4 of Procedure 5.1 and observe how the number of layers of the state changes as the head of $\mathcal{A}_k$ moves from o_{11} to the root in the sequence (5.1) of moves. There are four different cases:

1. A rewriting move is caused at a certain position. Let o be the innermost position among such positions. There are two different subcases:

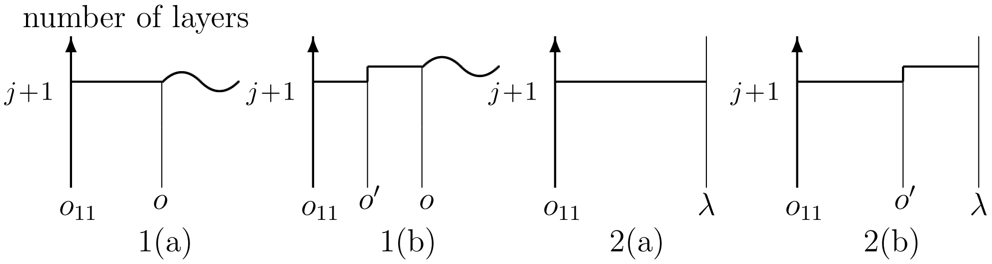

Figure 5: The number of layers of a state of $\mathcal{A}_k$ in the sequence (5.1).

   (a) The number of layers does not increase at any o' with $o \prec o' \prec o_{11}$.

   (b) There is a position o' with $o \prec o' \prec o_{11}$ such that the number of layers increases at o'.

2. There are no rewriting moves in the sequence. There are two subcases:

   (a) The number of layers does not increase at any o' with $\lambda \prec o' \prec o_{11}$.

   (b) There is a position o' with $\lambda \prec o' \prec o_{11}$ such that the number of layers increases at o'.

These four cases are illustrated in Fig. 5.

Assume that the number of layers changes as in case 1(a) above. In this case we can derive a contradiction as follows. First we assume a rewriting move at position o is proper and let $l' \to r'$ be the rewrite rule used for defining this transition rule in Step 4 of Procedure 5.1. Then, the state just before this rewriting move occurs at o can be written as $\langle r'\rho' \rangle$. Remark that $\text{layer}(\langle r'\rho' \rangle) = \text{layer}(p_{11}) \geq j+1$ since the number of layers does not change at any o' ($o \prec o' \prec o_{11}$). This implies that the $\mathcal{Q}_k$-substitution ρ' replaces a variable in r' with a state which has j or more layers (see the definition (3) of the number of layers). Therefore, by using the inductive hypothesis, the rule $l' \to r'$ must have rank j or more. On the other hand, the fact that the number of layers does not increase at o' with $o \prec o' \prec o_{11}$ implies that r' properly sticks out of l/o as follows.

Consider the moves of the TA from the position o_{11} to o. Since o is the inner most position among the positions where rewriting moves are caused, all moves at o' ($o \prec o' \preceq o_{11}$) are defined by **ADDTRANS**. By the construction of transition rules in **ADDTRANS**, it follows that the function symbol of l at the position $o \cdot o''$ is the same as the function symbol of r' at o for every o such that $o \cdot o'' \prec o_{11}$. Furthermore, it can be easily shown that when the head visits the position $o \cdot o''(o \cdot o'' \preceq o_{11})$ of l, the state $\langle r'\rho'/o'' \rangle$ is attached to that head. Thereby, at the variable position o_{11}, $\langle r'\rho'/o'' \rangle$ was attached

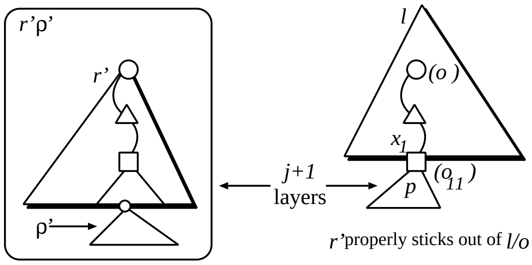

Figure 6: The number of layers and the sticking-out relation (Case 1(a)).

where o'' is such that $o \cdot o'' = o_{11}$, and this is the state p_{11}. Intuitively saying, the head goes up l along the path from o_{11} to o by changing the state from p_{11} to $\langle f(\ldots, p_{11}, \ldots)\rangle$ where f is the scanned symbol. This implies that r' properly sticks out of l/o by Case 1 of the definition of the sticking-out graph (Definition 4.1). We have observed that the rank of $l' \to r'$ is j or more, and thus the rank of $l \to r$ must be defined to be $j+1$ or more, a contradiction.

Next consider the case that the first rewriting transition rule used at o is defined in Case 3(ii) of **ADDTRANS** and let the rule be $p'' \to p$ such that $p'' = (p \setminus p_0) \cup p'$ and $p_0 \subseteq p$ for some rewriting transition rule $p' \to p_0$. The rewriting transition rule $p' \to p_0$ is either proper or non-proper. Assume $p' \to p_0$ is proper, then p' can be written as $\langle r'\rho'\rangle$ for some rewrite rule $l' \to r'$ and some $\mathcal{Q}_k$-substitution ρ'. From the fact that there is no rewriting move from o_{11} to o we can see that for every position o'' with $o \cdot o'' \preceq o_{11}$ when the head visits the position $o \cdot o''$ in l, a packed state which has $r'\rho'/o''$ as an element is attached to that head. Moreover from the construction of a non-ε rule whose right-hand side has more than one element (Case 3(iii) of **ADDTRANS**), the function symbol at $o \cdot o''$ ($o \cdot o'' \prec o_{11}$) in l coincides with the one in r' at o''. This implies that r' properly sticks out of l/o by Case 1 of the definition of sticking-out graph (Definition 4.1). By using this fact, we can derive a contradiction in the same way as in the case when the rule used at o is proper. Even if $p' \to p_0$ is non-proper, it is easy to see that there is a proper rewriting transition rule whose left-hand side is included in p'' and again a contradiction can be derived.

For other Cases 1(b), 2(a) and 2(b), we can derive a contradiction in a similar way (See the appendix). Thereby, it cannot happen that layer$(p_1) \geq j+1$ and the induction completes.

□

For an RL-FPO-TRS $\mathcal{R}$, the rank of every rule is less than $|\mathcal{R}|$ and hence the number of layers of any packed state is $|\mathcal{R}|$ or less by Lemma 5.7. By Lemma 5.6, the number of packed states is finite and the following theorem holds.

Theorem 5.9 *Procedure 5.1 halts for an RL-FPO-TRS.* □

In general, the running time of Procedure 5.1 is exponential to both of the size of a TRS $\mathcal{R}$ and the size of a TA $\mathcal{A}$.

Corollary 5.10 *LL-G-TRS^{-1} $\subset$ RL-GSM-TRS $\subset$ RL-FPO-TRS $\subset$ EPR-TRS.*

Proof. LL-G-TRS^{-1} $\subset$ RL-GSM-TRS can easily be shown by definition. RL-GSM-TRS $\subset$ RL-FPO-TRS is by Theorem 4.2. RL-FPO-TRS $\subset$ EPR-TRS is by Theorems 5.5 and 5.9. □

Corollary 5.10 implies that the conjecture in [7] is true, which says that right-linear semi-monadic term rewriting systems effectively preserve recognizability.

6 Decidable Approximations

In this section, we investigate decidable approximations of TRS along the lines of [4, 10, 12]. A TRS $\mathcal{R}'$ is an approximation of a TRS $\mathcal{R}$ if $\to^*_{\mathcal{R}} \subseteq \to^*_{\mathcal{R}'}$ and $\mathrm{NF}_{\mathcal{R}} = \mathrm{NF}_{\mathcal{R}'}$. An approximation mapping α is a mapping from TRSs to TRSs such that $\alpha(\mathcal{R})$ is an approximation of $\mathcal{R}$ for any TRS $\mathcal{R}$. For a class C of TRSs, a C approximation mapping is an approximation mapping such that $\alpha(\mathcal{R}) \in C$ for every TRS $\mathcal{R}$.

In 1996, Jacquemard[10] introduced a linear growing approximation mapping. Later Nagaya and Toyama[12] introduced a better approximation called a left-linear growing approximation mapping and presented decidable results on them. An RL-FPO^{-1}-TRS approximation mapping α is such that for a TRS $\mathcal{R}$, α replaces some variables in the right-hand side r_2 of a rewrite rule $l_2 \to r_2$ in $\mathcal{R}^{-1}$ with a new variable which is not in $\mathcal{V}ar(l_2)$, so that r_2 cannot contribute to an edge in the sticking-out graph of $\alpha(\mathcal{R}^{-1})$. For example, replacing variable x with x' in the right-hand side of the rule in R_2 of Example 4.1 yields an RL-FPO^{-1}-TRS approximation of $\mathcal{R}_2^{-1}$. The following results are a generalization of [12].

Let α be an approximation mapping and Ω be a fresh constant. A redex at a position o in $t \in \mathcal{T}(\mathcal{F})$ is α-needed if there exists no $s \in \mathrm{NF}_{\mathcal{R}}$ such that $t[o \leftarrow \Omega] \to^*_{\alpha(\mathcal{R})} s$ and s contains no Ω. This definition is due to [4]. If $\mathcal{R}$ is orthogonal, then every α-needed redex is a needed redex in the sense of Huet and Lévy [9]. Let $\text{CBN-NF}_{\alpha} = \{\mathcal{R} \mid$ every term $t \notin \mathrm{NF}_{\mathcal{R}}$ has an α-needed redex $\}$. By Theorems 15 and 29 in [4] and Lemma 2.1 of this paper, the following theorem holds.

Theorem 6.1 *Let $\mathcal{R}$ be a left-linear TRS and α be an EPR^{-1}-TRS approximation mapping. Then the following problems are decidable. (1) Is a given redex in a given term α-needed? (2) Is $\mathcal{R}$ in CBN-NF$_\alpha$?*

Corollary 6.2 *Let $\mathcal{R}$ be an orthogonal TRS in EPR^{-1}-TRS which satisfies the variable restriction such that l is not a variable and $\mathcal{V}ar(r) \subseteq \mathcal{V}ar(l)$ for every $l \to r \in \mathcal{R}$. (1) Every term $t \notin NF_{\mathcal{R}}$ has a needed redex. (2) It is decidable whether a given redex in a given term is needed.*

To conclude this section, we provide an orthogonal TRS $\mathcal{R}$ in FPO^{-1}-TRS such that there exists no left-linear growing approximation mapping β which satisfies $\mathcal{R} \in$ CBN-NF$_\beta$.

Example 6.1 Let $\mathcal{R} = \{g(h(x)) \to f(x,x,x)\} \cup \mathcal{R}'$ be an orthogonal TRS where $\mathcal{R}'$ consists of the following five rewrite rules:

$$f(a,b,x) \to a,\ f(b,x,a) \to a,\ f(x,a,b) \to a,$$
$$f(a,a,a) \to a,\ f(b,b,b) \to b.$$

It can be easily verified that $\mathcal{R}$ is in FPO^{-1}-TRS. Every term $t \notin$ NF$_{\mathcal{R}}$ has a needed redex in $\mathcal{R}$ by Corollary 5.10 and Corollary 6.2(1). On the other hand, a left-linear growing approximation mapping β should be $\beta(\mathcal{R}) = \{g(h(y)) \to f(x,x,x)\} \cup \mathcal{R}'$ for some variable $y \neq x$. Consider a term $t = f(g(h(a)), g(h(a)), g(h(a)))$. Obviously, $g(h(a)) \to^*_{\beta(\mathcal{R})} a$ and $g(h(a)) \to^*_{\beta(\mathcal{R})} b$. Hence, t has no β-needed redex. Thus, $\mathcal{R} \notin$ CBN-NF$_\beta$. □

7 Linearization

In this section, we prove the soundness lemma, Lemma 5.1, of Procedure 5.1. The procedure can accept some left-non-linear TRSs as an input. Dealing with non-linear terms is beyond the capability of TAs in general. Here we introduce a *linearization* of a non-left-linear TRS in order to deal with non-left-linear TRSs by TAs,

For an RL-TRS $\mathcal{R}$, let $n_{\mathcal{R}}$ be the smallest integer such that, for every rewrite rule $l \to r \in \mathcal{R}$, no variable occurs more than $n_{\mathcal{R}}$ times in l. Let $\wedge_{\mathcal{R}} = \{\wedge_i \mid 2 \leq i \leq n_{\mathcal{R}}\}$ be the set of new function symbols where the arity of $\wedge_i$ is i. Note that if $n_{\mathcal{R}} \leq 1$, then $\wedge_{\mathcal{R}} = \emptyset$ by definition. If the subscript i of the function symbol $\wedge_i$ is clear from the context, then we may write $\wedge$ instead of $\wedge_i$. Also we may write $\wedge$ instead of $\wedge_{\mathcal{R}}$. A term in $\mathcal{T}(\mathcal{F} \cup \wedge)$ is called a $\wedge$-*term*.

Definition 7.1 For an RL-TRS $\mathcal{R}$, α is a TRS which is defined as $\alpha = \{\wedge_n(x,\ldots,x) \to x \mid \wedge_n \in \wedge_{\mathcal{R}}, n \geq 2\}$. □

Example 7.1 Let $\mathcal{R}$ be $\{f(x,x) \to g(x)\}$, then $\alpha = \{\wedge_2(x,x) \to x\}$. For a term $s = g(\wedge(\wedge(a,a),a))$, $s \to_\alpha^* g(a)$. □

Definition 7.2 For an RL-TRS $\mathcal{R}$, a rewrite step $\to_{\mathcal{R}_\alpha}$ is the smallest relation on $\wedge$-terms containing the rewrite relation $\to_\mathcal{R}$ on $\mathcal{F}$-terms and closed under contexts on $\wedge$-terms. □

Definition 7.3 For a right-linear rewrite rule $l \to r$, the *conditional linearization* of $l \to r$ is a conditional rewrite rule defined as follows and written as $\wedge_L(l \to r)$: (1) Let $\mathcal{V}ar(l) = \{x_1, \ldots, x_n\}$. Assume x_i occurs at o_{ij} $(1 \leq j \leq \gamma_j)$ in l and if x_i occurs in r then it occurs at o_i. (2) Introduce new variables x_{ij} and y_i for $1 \leq i \leq n$ and $1 \leq j \leq \gamma_j$. (3) Define $\wedge_L(l \to r) = l[o_{ij} \leftarrow x_{ij} \mid 1 \leq i \leq n, 1 \leq j \leq \gamma_j] \to r[o_i \leftarrow \wedge(x_{i1}, \ldots, x_{i\gamma_i}) \mid$ for i such that x_i occurs in $r]$ with the condition $(x_{ij} = y_i \ (1 \leq i \leq n, 1 \leq j \leq \gamma_i))$. For an RL-TRS $\mathcal{R}$, define $\wedge(\mathcal{R}) = \{\wedge_L(l \to r) \mid l \to r \in \mathcal{R}\}$. □

Definition 7.4 For an RL-TRS $\mathcal{R}$, a rewrite step $\to_{\wedge(\mathcal{R})}$ is defined as follows: (1) $\to_{\wedge(\mathcal{R})} = \{(s,t) \mid (s,t) \in \to_{\wedge(\mathcal{R}),i}$ for some $i\}$. (2) $\to_{\wedge(\mathcal{R}),0} = \emptyset$. (3) $\to_{\wedge(\mathcal{R}),i+1} = \{(C[l\sigma], C[r\sigma]) \mid C$ is a context, $l \to r(x_1 = y_1, \ldots, x_n = y_n)$ is a conditional rewrite rule in $\wedge(\mathcal{R})$, σ is a substitution such that $y_i\sigma \in \mathcal{T}(\mathcal{F})$ for $1 \leq i \leq n$ and $x_i\sigma \ (\to_{\wedge(\mathcal{R}),i} \cup \to_\alpha)^* \ y_i\sigma$ $\}$ where the relation $(\to_{\wedge(\mathcal{R}),i} \cup \to_\alpha)^*$ is the reflexive and transitive closure of $\to_{\wedge(\mathcal{R}),i} \cup \to_\alpha$. We say $s \to_{\wedge(\mathcal{R}),i} t$ is a *rewrite step of degree* i. □

Definition 7.5 For two $\wedge$-terms s,t and an RL-TRS $\mathcal{R}$, (1) $s \to_{\alpha,\wedge(\mathcal{R})} t$ if $s \to_\alpha^* \cdot \to_{\wedge(\mathcal{R})} \cdot \to_\alpha^* t$, (2) $s \to_{\alpha,\mathcal{R}_\alpha} t$ if $s \to_\alpha^* \cdot \to_{\mathcal{R}_\alpha} \cdot \to_\alpha^* t$. □

In Definition 7.4, the reason why the domain of $y_i\sigma$ for $1 \leq i \leq n$ is restricted to $\mathcal{T}(F)$ is that if this condition is not assumed, then it may occur that, for two $\mathcal{F}$-terms s and t, $s \not\to_\mathcal{R}^* t$ but $s \to_{\wedge(\mathcal{R})}^* t$. For example, let $\mathcal{R} = \{f(x_1,x_1,x_2,x_2) \to g(x_1,x_2), g(x,x) \to c''\} \cup \mathcal{R}'$ where $\mathcal{R}' = \{a \to c, a \to c', b \to c, d \to c', e \to c', e \to c\}$ and consider two $\mathcal{F}$-terms $f(a,b,d,e)$ and c''. The conditional linearization of $\mathcal{R}$ is $\wedge(\mathcal{R}) = \{f(x_{11},x_{12},x_{21},x_{22}) \to g(\wedge(x_{11},x_{12}),\wedge(x_{21},x_{22}))\ (x_{11} = y_1, x_{12} = y_1, x_{21} = y_2, x_{22} = y_2), g(x_1,x_2) \to c''\ (x_1 = y, x_2 = y)\} \cup \mathcal{R}'$. If we ignore the condition that the domain of $y_i\sigma$ is restricted to $\mathcal{T}(\mathcal{F})$, then $f(a,b,d,e) \to_{\wedge(\mathcal{R})} g(\wedge(a,b),\wedge(d,e)) \to_{\wedge(\mathcal{R})}^* g(\wedge(c',c),\wedge(c',c)) \to_{\wedge(\mathcal{R})} c''$ holds. On the other hand, we can see that $f(a,b,d,e) \not\to_\mathcal{R}^* c''$.

Definition 7.6 For an RL-TRS $\mathcal{R}$, $\to_{\alpha,\mathcal{R}\cup\wedge(\mathcal{R})} = \to_{\alpha,\mathcal{R}_\alpha} \cup \to_{\alpha,\wedge(\mathcal{R})}$. □

For two terms s,t, if $s \to_1 \cdots \to_n t$ holds where $\to_i$ is either $\to_\alpha$ or $\to_{\mathcal{R}_\alpha}$ or $\to_{\wedge(\mathcal{R}),d_i}$, then $s \to_{\alpha,\mathcal{R}\cup\wedge(\mathcal{R})} t$ and we say that $\max\{d_i \mid$ for i such that $\to_i = \to_{\wedge(\mathcal{R}),d_i}$ $\}$ is the *maximum degree of* the sequence $s \to_1 \cdots \to_n t$.

Example 7.2 Let $\mathcal{R}_1 = \{f(x) \to g(x), h(x,x) \to h'(x)\}$, then we obtain $\wedge(\mathcal{R}_1) = \{f(x) \to g(x), h(x_1,x_2) \to h'(\wedge(x_1,x_2))\ (x_1 = y, x_2 = y)\}$. For a ground term $h(f(a),g(a))$, $h(f(a),g(a)) \to_{\alpha,\wedge(\mathcal{R}_1)} h'(\wedge(f(a),g(a))) \to^*_{\alpha,\wedge(\mathcal{R}_1)} h'(g(a))$. □

Lemma 7.1 *For a $\wedge$-term s, an $\mathcal{F}$-term t and an RL-TRS $\mathcal{R}$, $s \to^*_{\alpha,\mathcal{R}\cup\wedge(\mathcal{R})} t$ implies $s \to^*_{\alpha,\mathcal{R}} t$.*

Proof. The proof is shown by induction on the number of maximum degree of the rewrite sequence $s \to^*_{\alpha,\mathcal{R}\cup\wedge(\mathcal{R})} t$. For the basis, the lemma holds obviously. Assume the lemma holds for every sequence whose maximum degree of rewrite steps is $n-1$ or less and consider the case when the maximum degree is n. The inductive part is shown by another induction on the number of rewrite steps of degree n by $\to_{\wedge(\mathcal{R})}$. Assume the lemma holds for every sequence whose rewrite steps of degree n by $\to_{\wedge(\mathcal{R})}$ is $n'-1$ or less and consider the case for n'. A sequence which has n' rewrite steps of degree n by $\to_{\wedge(\mathcal{R})}$ can be written as:

$$\begin{array}{rcl} s & \to^*_{\alpha,\mathcal{R}\cup\wedge(\mathcal{R})} & s' \\ & = & s'[o \leftarrow l\sigma] \\ & \to_{\wedge(\mathcal{R})} & s'[o \leftarrow r\sigma] \\ & = & t' \\ & \to^*_{\alpha,\mathcal{R}\cup\wedge(\mathcal{R})} & t \end{array} \qquad (7.1)$$

where s', t' are $\wedge$-terms, $s'[o \leftarrow l\sigma] \to_{\wedge(\mathcal{R})} s'[o \leftarrow r\sigma]$ is the first rewrite step of degree n by $\to_{\wedge(\mathcal{R})}$, $l \to r$ is a rewrite rule in $\wedge(\mathcal{R})$, σ is a substitution and o is a position in s'. Remark that the sequence $s'[o \leftarrow r\sigma] \to^*_{\alpha,\mathcal{R}\cup\wedge(\mathcal{R})} t$ contains $n'-1$ rewrite steps of degree n by $\to_{\wedge(\mathcal{R})}$. We assume the following:

1. $\wedge_L(l' \to r') = l \to r$ where $l' \to r' \in \mathcal{R}$.
2. l' has m variables $x_1, \ldots x_m$.
3. For $1 \le i \le m$, x_i occurs at positions o_{ij} $(1 \le i \le \gamma_i)$ in l' and if x_i occurs in r' then it occurs at o_i.
4. For $1 \le i \le m$ and $1 \le i \le \gamma_i$, $l/o_{ij} = x_{ij}$, which is a new variable for defining $l \to r$ from $l' \to r'$ in Definition 7.3.
5. $\sigma = \{x_{ij} \mapsto t_{ij} \mid 1 \le i \le m, 1 \le j \le \gamma_i\}$.

In the following, we define an $\mathcal{F}$-term t_k for each x_k ($1 \leq k \leq m$).

If x_k does not occur in r', then from the definition of $\rightarrow_{\wedge(\mathcal{R})}$ there exists an $\mathcal{F}$-term t_k such that $t_{kj} \rightarrow^*_{\alpha,\wedge(\mathcal{R})} t_k$ for $1 \leq j \leq \gamma_k$ where the degree of each rewrite step $\rightarrow_{\wedge(\mathcal{R})}$ is less than or equal to $n-1$. By the inductive hypothesis for n, we obtain

$$t_{kj} \rightarrow^*_{\alpha,\mathcal{R}} t_k \quad (1 \leq j \leq \gamma_k). \tag{7.2}$$

If x_k occurs in r', then $r\sigma/o_k = \wedge(t_{k1}, \ldots, t_{k\gamma_k})$. Consider how the subterm $\wedge(t_{k1}, \ldots, t_{k\gamma_k})$ of $t = s'[o \leftarrow r\sigma]$ is rewritten in the rewrite sequence (7.1). Since t' is rewritten to a term t in $\mathcal{T}(\mathcal{F})$ (i.e., all the $\wedge$ symbols disappear during the rewriting), there are two cases.

1. The subterms $t_{k1}, \ldots, t_{k\gamma_k}$ of $\wedge(t_{k1}, \ldots, t_{k\gamma_k})$ are rewritten to an identical term t_k in $\mathcal{T}(\mathcal{F})$, i.e. $t_{kj} \rightarrow^*_{\alpha,\mathcal{R}\cup\wedge(\mathcal{R})} t_k$ for $1 \leq j \leq \gamma_k$. By applying the inductive hypothesis for n' (when $n' \geq 2$) or the inductive hypothesis for n (when $n' = 1$) to these rewrite sequences, we obtain the relations

$$t_{kj} \rightarrow^*_{\alpha,\mathcal{R}} t_k \quad (1 \leq j \leq \gamma_k). \tag{7.3}$$

2. The term $\wedge(t_{k1}, \ldots, t_{k\gamma_k})$ is rewritten to $\wedge(t'_{k1}, \ldots, t'_{k\gamma_k})$ and disappear in the subsequent rewrite steps, i.e.,

$$\begin{array}{rcl} t' & = & s'[o \leftarrow r\sigma[o_k \leftarrow \wedge(t_{k1}, \ldots, t_{k\gamma_k})]] \\ & \rightarrow^*_{\alpha,\mathcal{R}\cup\wedge(\mathcal{R})} & t'' \\ & = & t''[o' \leftarrow l''\sigma''] \\ & = & t''[o' \leftarrow l''\sigma''[o'_1 \leftarrow \wedge(t'_{k1}, \ldots, t'_{k\gamma_k})]] \qquad (7.4) \\ & \rightarrow_{\alpha,\mathcal{R}\cup\wedge(\mathcal{R})} & t''[o' \leftarrow r''\sigma''] \qquad (7.5) \\ & \rightarrow^*_{\alpha,\mathcal{R}\cup\wedge(\mathcal{R})} & t \end{array}$$

where t'' is a $\wedge$-term, o' is a position in t'', $l'' \rightarrow r'' \in \wedge(\mathcal{R})$, σ'' is a substitution, o'_1 is a position in $l''\sigma''$ and there is a variable position o''_1 in l'' such that $o''_1 \leq o'_1$ and the variable l''/o''_1 does not occur in r''. The rewrite rule $l'' \rightarrow r''$ must be in $\wedge(\mathcal{R})$ since the co-domain of σ'' contains function symbols in $\wedge$. The position of the subterm $\wedge(t_{k1}, \ldots, t_{k\gamma_k})$ in t' is $o \cdot o_k$. By the rewrite step (7.5) and the definition of $\rightarrow_{\wedge(\mathcal{R})}$, there is an $\mathcal{F}$-term $t_k \in \mathcal{T}(\mathcal{F})$ such that $t'_{kj} \rightarrow^*_{\alpha,\wedge(\mathcal{R})} t_k$ which has only rewrite steps by $\rightarrow_{\wedge(\mathcal{R})}$ of degree $n-1$ or less for $1 \leq j \leq \gamma_k$ by Definition 7.4. By the inductive hypothesis for n, we obtain

$$t'_{kj} \rightarrow^*_{\alpha,\mathcal{R}} t_k. \tag{7.6}$$

Also from the sequence (7.4), it follows that

$$t_{kj} \quad \rightarrow^*_{\alpha,\mathcal{R}\cup\wedge(\mathcal{R})} \quad t'_{kj} \quad (1 \leq j \leq \gamma_k). \tag{7.7}$$

From (7.6) and (7.7), we obtain the sequences

$$t_{kj} \quad \rightarrow^*_{\alpha,\mathcal{R}\cup\wedge(\mathcal{R})} \quad t_k \quad (1 \leq j \leq \gamma_k) \tag{7.8}$$

where the numbers of rewrite steps of degree n by $\rightarrow_{\wedge(\mathcal{R})}$ are less than or equal to $n' - 1$. By the inductive hypothesis,

$$t_{kj} \quad \rightarrow^*_{\alpha,\mathcal{R}} \quad t_k \quad (1 \leq j \leq \gamma_k). \tag{7.9}$$

It follows from (7.2), (7.3) and (7.9) that

$$\begin{aligned} s' &= s'[o \leftarrow l\sigma] \\ &= s'[o \leftarrow l[o_{ij} \leftarrow t_{ij} \mid 1 \leq i \leq m, 1 \leq j \leq \gamma_i]] \\ &\rightarrow^*_{\alpha,\mathcal{R}} s'[o \leftarrow l'\sigma'] \end{aligned} \tag{7.10}$$

where $\sigma' = \{x_i \mapsto t_i \mid 1 \leq i \leq m\}$. Since $l' \rightarrow r' \in \mathcal{R}$ and $t_i \in \mathcal{T}(\mathcal{F})$ $(1 \leq i \leq m)$, we obtain

$$s'[o \leftarrow l'\sigma'] \rightarrow_{\mathcal{R}_\alpha} s'[o \leftarrow r'\sigma'] \tag{7.11}$$

by Definition 7.2. Since $s'[o \leftarrow r\sigma] \rightarrow^*_{\alpha,\mathcal{R}\cup\wedge(\mathcal{R})} t$ and there is no function symbol $\wedge$ in the left-hand side of any rewrite rule in $\mathcal{R} \cup \wedge(\mathcal{R})$,

$$s'[o \leftarrow r'\sigma'] \rightarrow^*_{\alpha,\mathcal{R}\cup\wedge(\mathcal{R})} t, \tag{7.12}$$

which contains $n' - 1$ or less rewrite steps of degree n by $\rightarrow_{\wedge(\mathcal{R})}$. By the relations (7.10),(7.11) and (7.12), we obtain $s \rightarrow^*_{\alpha,\mathcal{R}\cup\wedge(\mathcal{R})} t$ where the number of rewrite steps of degree n by $\rightarrow_{\wedge(\mathcal{R})}$ is less than or equal to $n' - 1$. By the inductive hypothesis, we obtain $s \rightarrow^*_{\alpha,\mathcal{R}} t$ and the lemma holds. □

7.1 Proof of the soundness lemma

Before proving the soundness of Procedure 5.1, we need some notions concerning with the TA constructed in the procedure.

A state q in $\mathcal{Q}_k$ is *singleton* if $|q| = 1$. A transition rule in Δ_k is *singleton* if its right-hand side is a singleton state. A move caused by a singleton transition rule is called a *singleton move*. For a TA $\mathcal{A}_k$ and a state $q \in \mathcal{Q}_k$, let $\mathcal{A}_{k^-}(q)$ (resp. $\mathcal{A}_{k^\bullet}(q)$) be the TA obtained from $\mathcal{A}_k(q)$ by removing every rewriting transition rules (resp. non-singleton transition rules). For an $\mathcal{F}$-term s and a state $q \in \mathcal{Q}_k$, if $s \vdash_k^* q$ without any rewriting moves (resp. non-singleton moves), then we write $s \vdash_{k^-}^* q$ (resp. $s \vdash_{k^\bullet}^* q$). Remark that if $s \vdash_{k^\bullet}^* q$, then the move does not contain any non-proper rewriting moves since every non-proper rewriting transition rule is non-sigleton (see Case 3(ii) of **ADDTRANS**).

For a set $\wedge$ and a TA $\mathcal{A} = (\mathcal{F}, \mathcal{Q}, \mathcal{Q}_{\mathit{final}}, \Delta)$, the extended TA $\wedge(\mathcal{A})$ for $\mathcal{T}(\mathcal{F} \cup \wedge)$ is defined as $\wedge(\mathcal{A}) = (\mathcal{F} \cup \wedge, \mathcal{Q}, \mathcal{Q}_{\mathit{final}}, \Delta \cup \Delta_\wedge)$ where $\Delta_\wedge = \{\wedge_n(q_1, \ldots, q_n) \to q, \wedge_n(q_1, \ldots, q_n) \to \langle t \rangle \mid \wedge_n \in \wedge, q_1, \ldots, q_n \in \mathcal{Q}, q_1 \cup \cdots \cup q_n = q \in \mathcal{Q}, t \in q\}$. A move caused by a transition rule in $\Delta_\wedge$ is called a $\wedge$-move. For a TA $\mathcal{A}_k$ in the procedure, we write $\vdash_{\wedge,k}$ instead of $\vdash_{\wedge(\mathcal{A}_k)}$. The TAs $\mathcal{A}_{\wedge,k^-}$, $\mathcal{A}_{\wedge,k^\bullet}$ and the relations $\vdash_{\wedge,k^-}, \vdash_{\wedge,k^\bullet}$ are similarly defined and we will use their combinations, e.g. $\vdash_{\wedge,k^{-,\bullet}}$.

Lemma 7.2 *Let $r \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ be a linear term with m variables $x_1, \ldots, x_m$ at $o_1, \ldots, o_m$, respectively, s be a $\wedge$-term and ρ be a substitution $\{x_i \mapsto q_i \mid 1 \le i \le m\}$ where $q_i \in \mathcal{Q}_k$ $(1 \le i \le m)$. If r is a variable, then let $t_{r\rho} = r\rho$. Otherwise, let $t_{r\rho} = \langle r\rho \rangle$. If $t_{r\rho} \in \mathcal{Q}_k$ and $s \vdash_{\wedge,k^-}^* t_{r\rho}$, then the sequence can be written as $s \vdash_{\wedge,k^-}^* s[o_i \leftarrow q_i \mid 1 \le i \le m] \vdash_{\wedge,k^-}^* t_{r\rho}$.* □

The next lemma states that if a $\wedge$-term s is accepted by a state q then there is a $\wedge$-term s' such that s' is accepted by q only with singleton moves and s' can be rewritten to s by rewrite rule $\wedge_i(x, \ldots, x) \to x$ $(i \ge 2)$.

Lemma 7.3 *For a $\wedge$-term s and a state $q \in \mathcal{Q}_k$, if $s \vdash_{\wedge,k}^* q$, then there is a $\wedge$-term s' such that $s' \vdash_{\wedge,k^\bullet}^* q$ and $s' \to_\alpha^* s$.*

Proof. The proof is shown by induction on the number of non-singleton rewriting moves in the sequence β: $s \vdash_{\wedge,k}^* q$. For the base case, the lemma holds obviously. Assume the lemma holds for every sequence which has at most $n-1$ non-singleton rewriting moves and consider the case for n. In this case, β can be written as

$$s \vdash_{\wedge,k^\bullet}^* s[o \leftarrow t] \vdash_k s[o \leftarrow p] \vdash_{\wedge,k}^* q \tag{7.13}$$

where o is in $\mathcal{P}os(s)$, $t \to p$ is a non-singleton transition rule and the move $s[o \leftarrow t] \vdash_k s[o \leftarrow p]$ is the first non-singleton rewriting move in β. Assume $p = \langle t_1, \ldots, t_m \rangle$ $(m \ge 2)$. There are three cases (1),(2) and (3) for the transition rule $t \to p$:

(1) If t is of the form $f(p_1, \ldots, p_{a(f)})$ where $f \in \mathcal{F}$ and $p_i \in \mathcal{Q}_k$ for $1 \le i \le a(f)$, transition rules $t \to \langle t_i \rangle$ for $1 \le i \le m$ are also defined in Case 3(iii) of **ADDTRANS**.

(2) If $t \to p$ is a proper rewriting transition rule, then $t \to \langle t_i \rangle$ for $1 \leq i \leq m$ are also defined in Step 4(b) of Procedure 5.1.

Let $s'' = s[o \leftarrow \wedge_m(s/o, \ldots, s/o)]$, then in both cases (1) and (2), we have

$$\begin{aligned} s'' &\vdash^*_{\wedge,k^\bullet} s[o \leftarrow \wedge_m(t, \ldots, t)] \\ &\vdash^*_{k^\bullet} s[o \leftarrow \wedge_m(\langle t_1 \rangle, \ldots, \langle t_m \rangle)] \\ &\vdash_\wedge s[o \leftarrow \bigcup_{1 \leq i \leq m} \langle t_i \rangle] \\ &= s[o \leftarrow p] \\ &\vdash^*_{\wedge,k} q. \end{aligned} \tag{7.14}$$

(3) If $t \in \mathcal{Q}_k$ and the transition rule $t \to p$ is defined in Case 3(ii) of **ADDTRANS**, there are two cases: Assume that the transition rule $t \to p$ is defined from a transition rule $p' \to p_0$ such that $t = (p \backslash p_0) \cup p'$.

(3a) If $p' \to p_0$ is a singleton transition rule, i.e. $|p_0| = 1$, or $p' \to p_0$ is not a rewriting transition rule, then it is easy to see that there is a singleton transition rule $q' \to q_0$ such that $t = (p \setminus q_0) \cup q'$. (Especially, $q' = p'$ and $q_0 = p_0$ in the former case.) Assume $q' = \langle t'_1, \ldots, t'_{m'} \rangle$ and $q_0 = \langle t_1 \rangle$ without loss of generality. Let $s'' = s[o \leftarrow \wedge_m(\wedge_{m'}(s/o, \ldots, s/o), s/o, \ldots, s/o)]$, then we have

$$\begin{aligned} s'' &\vdash^*_{\wedge,k^\bullet} s[o \leftarrow \wedge_m(\wedge_{m'}(t, \ldots, t), t, \ldots, t)] \\ &\vdash^*_{\wedge,k^\bullet} s[o \leftarrow \wedge_m(\wedge_{m'}(\langle t'_1 \rangle, \ldots, \langle t'_{m'} \rangle), \langle t_2 \rangle, \ldots, \langle t_m \rangle)] \\ &\qquad \text{(by Case 3(i) of } \textbf{ADDTRANS}) \\ &\vdash_\wedge s[o \leftarrow \wedge_m(\bigcup_{1 \leq i \leq m'} \langle t'_i \rangle, \langle t_2 \rangle, \ldots, \langle t_m \rangle)] \\ &= s[o \leftarrow \wedge_m(q', \langle t_2 \rangle, \ldots, \langle t_m \rangle)] \\ &\vdash_{\wedge,k^\bullet} s[o \leftarrow \wedge_m(q_0, \langle t_2 \rangle, \ldots, \langle t_m \rangle)] \\ &= s[o \leftarrow \wedge_m(\langle t_1 \rangle, \langle t_2 \rangle, \ldots, \langle t_m \rangle)] \\ &\vdash_\wedge s[o \leftarrow \bigcup_{1 \leq i \leq m} \langle t_i \rangle] \\ &= s[o \leftarrow p] \\ &\vdash^*_{\wedge,k} q. \end{aligned} \tag{7.15}$$

Since there are at most $n - 1$ non-singleton rewriting moves in both (7.14) and (7.15), by the inductive hypothesis, there is a term s' such that $s' \vdash^*_{\wedge,k^\bullet} q$ and $s' \to^*_\alpha s''$. Obviously, $s'' \to^*_\alpha s$ and the lemma holds.

(3b) If $p' \to p_0$ is a rewriting transition rule and $|p_0| \geq 2$, then there are two cases: Assume $p' = \langle t'_1, \ldots, t'_{m'} \rangle$ and $p \setminus p_0 = \langle t_{j_1}, \ldots, t_{j_{m''}} \rangle$ where $m'' = |p| - |p_0|$.

(i) If $|p'| = 1$ (i.e., $p' = \langle t'_1 \rangle$), then for $s''' = s[o \leftarrow \wedge_{m''+1}(s/o, \ldots, s/o)]$ we have

$$\begin{aligned} s''' &\vdash^*_{\wedge,k^\bullet} s[o \leftarrow \wedge_{m''+1}(\langle t'_1 \rangle, \langle t_{j_1} \rangle \ldots, \langle t_{j_{m''}} \rangle)] \\ &\vdash_k s[o \leftarrow \wedge_{m''+1}(p_0, \langle t_{j_1} \rangle \ldots, \langle t_{j_{m''}} \rangle)] \\ &\vdash_\wedge s[o \leftarrow \bigcup_{1 \leq i \leq m''} \langle t_{j_i} \rangle \cup p_0] \\ &= s[o \leftarrow p] \\ &\vdash^*_{\wedge,k} q. \end{aligned} \tag{7.16}$$

(ii) If $|p'| > 1$, then let

$$s''' = s[o \leftarrow \wedge_{m''+1}(\wedge_{m'}(s/o, \ldots, s/o), s/o, \ldots, s/o)].$$

We have

$$\begin{aligned} s''' &\vdash^*_{\wedge,k^\bullet} s[o \leftarrow \wedge_{m''+1}(\wedge_{m'}(\langle t'_1 \rangle, \ldots, \langle t'_{m'} \rangle), \langle t_{j_1} \rangle, \ldots, \langle t_{j_{m''}} \rangle)] \\ &\vdash_\wedge s[o \leftarrow \wedge_{m''+1}(\bigcup_{1 \leq i \leq m'} \langle t'_i \rangle, \langle t_{j_1} \rangle, \ldots, \langle t_{j_{m''}} \rangle)] \\ &= s[o \leftarrow \wedge_{m''+1}(p', \langle t_{j_1} \rangle, \ldots, \langle t_{j_{m''}} \rangle)] \\ &\vdash_k s[o \leftarrow \wedge_{m''+1}(p_0, \langle t_{j_1} \rangle, \ldots, \langle t_{j_{m''}} \rangle)] \\ &\vdash_\wedge s[o \leftarrow \bigcup_{1 \leq i \leq m''} \langle t_{j_i} \rangle \cup p_0] \\ &= s[o \leftarrow p] \\ &\vdash^*_{\wedge,k} q. \end{aligned} \tag{7.17}$$

In both cases (i) and (ii) of (3b), $s''' \to^*_\alpha s$ holds. If $p' \to p_0$ is a proper rewriting transition rule, then both sequences of moves (7.16) and (7.17) are of the form in case (2) in this proof. Otherwise, repeating the same discussion of this case (3), we can finally obtain a proper rewriting transition rule and a sequence of of the form in case (2). Therefore, we can show that there is a term s'' for s''' in the moves such that $s'' \vdash^*_{\wedge,k} q$ which has at most $n-1$ non-singleton rewriting moves and $s'' \to^*_\alpha s'''$. Thus the lemma holds by the inductive hypothesis. □

Definition 7.7 For a $\wedge$-term s and an RL-TRS $\mathcal{R}$, $\mathcal{E}_{\alpha,\wedge(\mathcal{R})}(s)$ is true if and only if there is an $\mathcal{F}$-term s' such that $s \to^*_{\alpha,\wedge(\mathcal{R})} s'$. □

Example 7.3 Consider the TRS $\mathcal{R}_1$ in Example 7.2. Let a term $s = h'(\wedge(f(a), g(a)))$, then $\mathcal{E}_{\alpha,\wedge(\mathcal{R}_1)}(s)$ is true. On the other hand, let $s' = h'(\wedge(f(a),\ g(c)))$, then $\mathcal{E}_{\alpha,\wedge(\mathcal{R}_1)}(s')$ is false. □

Lemma 7.4 *For $\wedge$-terms s, s' and an RL-TRS $\mathcal{R}$ such that $\mathcal{E}_{\alpha,\wedge(\mathcal{R})}(s)$ is true, if s' is a subterm of s or $s \to_\alpha^* s'$, then $\mathcal{E}_{\alpha,\wedge(\mathcal{R})}(s')$.* □

Lemma 7.5 *For a $\wedge$-term s and an RL-TRS $\mathcal{R}$ such that $\mathcal{E}_{\alpha,\wedge(\mathcal{R})}(s)$ is true, and for a state $q \in \mathcal{Q}_k$, if $s \vdash_{\wedge,k}^* q$, then there exists a $\wedge$-term u' such that $\mathcal{E}_{\alpha,\wedge(\mathcal{R})}(u')$ is true, $u' \to_{\alpha,\wedge(\mathcal{R})}^* s$ and $u' \vdash_{\wedge,k^-,\bullet}^* q$.*

Proof. By Lemma 7.3, there is a $\wedge$-term s_α such that $s_\alpha \to_\alpha^* s$ and

$$s_\alpha \vdash_{\wedge,k^\bullet}^* q. \tag{7.18}$$

The proof is shown by induction on the maximum degree of rewriting moves in the sequence (7.18). For the basis, let $u' = s_\alpha$ and the lemma holds. Assume the lemma holds for every sequence where the maximum degree of rewriting moves is $k-1$ or less and consider the case for k (≥ 1). The inductive part is shown by another induction on the number of rewriting moves of degree k in the sequence (7.18). Assume the lemma holds for every sequences which has $n-1$ rewriting moves of degree k and consider the case for n. The sequence (7.18) which has n rewriting moves of degree k can be written as

$$s_\alpha \vdash_{\wedge,k^\bullet}^* s_\alpha[o \leftarrow q'] \vdash_{k^\bullet} s_\alpha[o \leftarrow q''] \vdash_{\wedge,k^\bullet}^* q$$

where o is a position in s and the move $s_\alpha[o \leftarrow q'] \vdash_{k^\bullet} s_\alpha[o \leftarrow q'']$ is the first rewriting move of degree k. Remark that the transition rule used in this move is a proper rewriting move since every singleton rewriting transition rule is proper. Also note that $s_\alpha[o \leftarrow q''] \vdash_{\wedge,k^\bullet}^* q$ contains only $n-1$ rewriting moves of degree k. By the definition of TAs, $s_\alpha/o \vdash_{\wedge,k^\bullet}^* q' \vdash_{k^\bullet} q''$. There is no rewriting move of degree k in $s_\alpha/o \vdash_{\wedge,k^\bullet}^* q'$. By the inductive hypothesis on k, there is a term v such that $\mathcal{E}_{\alpha,\wedge(\mathcal{R})}(v)$ is true, $v \to_{\alpha,\wedge(\mathcal{R})}^* s_\alpha/o$ and $v \vdash_{\wedge,k^-,\bullet}^* q'$.

For the sequence β: $v \vdash_{\wedge,k^-,\bullet}^* q' \vdash_{k^\bullet} q''$, without loss of generality, assume that

1. $q' \to q''$ used in the last move in β is defined for a rewrite rule $l \to r \in \mathcal{R}$,
2. l has m variables $x_1, \ldots, x_m$,
3. the variable x_i has γ_i positions in l at $o_{ij} \in \mathcal{P}os(l)$ $(1 \leq j \leq \gamma_i)$, and
4. if the variable x_i occurs in r, then it occurs at $o_i \in \mathcal{P}os(r)$.

Let $l' \to r'$ be $\wedge_L(l \to r)$. Since the last move $q' \vdash_{k^\bullet} q''$ is a rewriting move of degree k, and since it is defined for the rule $l \to r$ at Step 4 of Procedure 5.1, there are states p_{ij} $(1 \le i \le m, 1 \le j \le \gamma_i)$ and q''_0 in $\mathcal{Q}_{k-1}$ such that

$$l[o_{ij} \leftarrow p_{ij} \mid 1 \le i \le m, 1 \le j \le \gamma_i] \vdash^*_{k-1} q''_0, \tag{7.19}$$

$$\mathcal{L}_{p_{i1}}(\mathcal{A}_{k-1}) \cap \cdots \cap \mathcal{L}_{p_{i\gamma_i}}(\mathcal{A}_{k-1}) \neq \emptyset \tag{7.20}$$

where $q'' = q''_0$ or $q'' = \langle t \rangle$ for some $t \in q''_0$. Furthermore, for the substitution $\rho = \{x_i \mapsto p_i \mid \text{for } i \text{ such that } x_i \text{ occurs in } r\}$ where $p_i = \bigcup_{1 \le j \le \gamma_i} p_{ij}$, if $r \in \mathcal{V}$ then $q' = r\rho$ else $q' = \langle r\rho \rangle$. By Lemma 7.2, we can write the sequence $v \vdash^*_{\wedge,k^-,\bullet} q'$ as

$$v \vdash^*_{\wedge,k^-,\bullet} v[o_i \leftarrow p_i \mid \text{for } i \text{ such that } x_i \text{ occurs in } r] \vdash^*_{\wedge,k^-,\bullet} q'. \tag{7.21}$$

Define substitutions σ and σ' as $\sigma = \{x_i \mapsto u_i \mid 1 \le i \le m\}$ and $\sigma' = \{x_{ij} \mapsto u_{ij} \mid 1 \le i \le m, 1 \le j \le \gamma_i\}$ where u_i and u_{ij} with $1 \le i \le m$ and $1 \le j \le \gamma_i$ are defined as follows:

1. If x_i occurs in r, then let $u_i = v/o_i$. By (7.21) we have $u_i \vdash^*_{\wedge,k^-,\bullet} p_i$. If $\gamma_i > 1$, then the sequence can be written as $u_i \vdash^*_{\wedge,k^-,\bullet} \wedge(p_{i1}, \ldots, p_{i\gamma_i}) \vdash_\wedge p_i$. In this case, let $u_{ij} = u_i/j$ for $1 \le i \le m, 1 \le j \le \gamma_i$. If $\gamma_i = 1$, then let $u_{i1} = u_i$. Remark that $r\sigma = r'\sigma'$. Also, $u_{ij} \vdash^*_{\wedge,k^-,\bullet} p_{ij}$ holds for $1 \le j \le \gamma_i$ since $u_i \vdash^*_{\wedge,k^-,\bullet} p_i$. By the fact that $\mathcal{E}_{\alpha,\wedge(\mathcal{R})}(v)$ is true and by Lemma 7.4,
$$\mathcal{E}_{\alpha,\wedge(\mathcal{R})}(u_i) \text{ is true.} \tag{7.22}$$

2. If x_i does not occur in r, then u_i is chosen to satisfy $u_i \in \mathcal{L}_{p_{i1}}(\mathcal{A}_{k-1}) \cap \cdots \cap \mathcal{L}_{p_{i\gamma_i}}(\mathcal{A}_{k-1})$, that is $u_i \vdash^*_{k-1} p_{ij}$ $(1 \le j \le \gamma_i)$. Such u_i exists by (7.20) and can be found effectively. By the inductive hypothesis on k, there are terms u_{ij} for $1 \le j \le \gamma_i$ such that $u_{ij} \to^*_{\alpha,\wedge(\mathcal{R})} u_i$ and
$$u_{ij} \vdash^*_{\wedge,k^-,\bullet} p_{ij}. \tag{7.23}$$

In either case 1 or 2, we have

$$u_{ij} \vdash^*_{\wedge,k^-,\bullet} p_{ij} \quad (1 \le j \le \gamma_i). \tag{7.24}$$

Let $v' = l'\sigma'$, then by (7.19) and (7.24), we have

$$\begin{aligned} v' &= l'\sigma' \\ &\vdash^*_{\wedge,k^-,\bullet} l'[o_{ij} \leftarrow p_{ij} \mid 1 \le i \le m, 1 \le j \le \gamma_i] \\ &\vdash^*_{k-1} q''_0. \end{aligned}$$

In either case $q'' = q''_0$ or $q'' = \langle t \rangle$ for some $t \in q''_0$, we have $v' \vdash^*_{\wedge,k-1} q''$. On the other hand, by (7.22), (7.23) and the fact that $(\rightarrow_{\wedge(\mathcal{R}),i} \cup \rightarrow_\alpha)^* \subseteq \rightarrow^*_{\alpha,\wedge(\mathcal{R})}$ for any i,

$$v' = l'\sigma' \rightarrow_{\wedge(\mathcal{R})} r'\sigma' = r\sigma = v. \tag{7.25}$$

That is, $v' \rightarrow_{\wedge(\mathcal{R})} v$. By the definition of TAs and the discussions above, we obtain

$$s_\alpha[o \leftarrow v'] \rightarrow_{\alpha,\wedge(\mathcal{R})} s_\alpha[o \leftarrow v] \rightarrow^*_{\alpha,\wedge(\mathcal{R})} s_\alpha[o \leftarrow s_\alpha/o] = s_\alpha \tag{7.26}$$

and $s_\alpha[o \leftarrow v'] \vdash^*_{\wedge,k-1} s_\alpha[o \leftarrow q''] \vdash^*_{\wedge,k\bullet} q$ where $s[o \leftarrow q''] \vdash^*_{\wedge,k\bullet} q$ contains $n-1$ rewriting moves of degree k. By the inductive hypothesis on the number of rewriting moves of degree k (when $n > 1$) or on the maximum degree of rewriting moves (when $n = 1$), there is a $\wedge$-term u' such that $\mathcal{E}_{\alpha,\wedge(\mathcal{R})}(u')$ is true,

$$u' \rightarrow^*_{\alpha,\wedge(\mathcal{R})} s_\alpha[o \leftarrow v'], \tag{7.27}$$

and

$$u' \vdash^*_{\wedge,k^-,\bullet} q. \tag{7.28}$$

By (7.26), (7.27) and the fact that $s_\alpha \rightarrow^*_\alpha s$, we obtain

$$u' \rightarrow^*_{\alpha,\wedge(\mathcal{R})} s. \tag{7.29}$$

By (7.28) and (7.29), the lemma holds. □

Lemma 7.6 *For an $\mathcal{F}$-term s, an RL-TRS $\mathcal{R}$ and a state $q \in \mathcal{Q}_k$, if $s \vdash^*_k q$, then there exists an $\mathcal{F}$-term u such that $u \rightarrow^*_{\mathcal{R}} s$ and $u \vdash^*_{k^-} q$.*

Proof. Suppose $s \vdash^*_k q$ for an $\mathcal{F}$-term $s \in \mathcal{T}(\mathcal{F})$ and $q \in \mathcal{Q}_k$ with $|q| = 1$. By Lemma 7.5 and the fact that $\mathcal{E}_{\alpha,\wedge(\mathcal{R})}(s)$, there is a $\wedge$-term u' such that $\mathcal{E}_{\alpha,\wedge(\mathcal{R})}(u')$ is true, $u' \rightarrow^*_{\alpha,\wedge(\mathcal{R})} s$ and $u' \vdash^*_{\wedge,k^-,\bullet} q$. By Lemma 7.1 and the fact $s \in \mathcal{T}(\mathcal{F})$, we obtain $u' \rightarrow^*_{\alpha,\mathcal{R}} s$. In the following, we construct from u' an $\mathcal{F}$-term u such that $u \rightarrow^*_{\mathcal{R}} s$ and $u \vdash^*_{k^-} q$ by replacing every subterm of the form $\wedge_m(t_1, \ldots, t_m)$ where $t_i \in \mathcal{T}(\mathcal{F})$ $(1 \le i \le m)$ with some term in $\{t_i \mid 1 \le i \le m\}$ from the leaves to the root. Assume $u'/o = \wedge_m(t_1, \ldots, t_m)$ and $t_i \in \mathcal{T}(\mathcal{F})$ for $1 \le i \le m$. Since $t_i \in \mathcal{T}(\mathcal{F})$ for $1 \le i \le m$ and all moves in the sequence $u' \vdash^*_{\wedge,k^-,\bullet} q$ are singleton, we have

$$\begin{aligned} u' \quad & \vdash^*_{k^-,\bullet} \quad u'[o \leftarrow \wedge_m(\langle t'_1 \rangle, \ldots, \langle t'_m \rangle)] \\ & \vdash_\wedge \quad u'[o \leftarrow \langle t'_1, \ldots, t'_m \rangle] \\ & \vdash^*_{\wedge,k^-,\bullet} \quad q \end{aligned} \tag{7.30}$$

where $\langle t'_i \rangle \in \mathcal{Q}_k$. Let $t \to \langle t' \rangle \in \Delta_k$ be the transition rule which is used to consume the state $\langle t'_1, \ldots, t'_m \rangle$ in v in the subsequence of (7.30) from $u'[o \leftarrow \langle t'_1, \ldots, t'_m \rangle]$ to q. There are two cases for t: (1) $t = \langle t'_1, \ldots, t'_m \rangle$ and $t' \in t$ and (2) t is of the form $f(p_1, \ldots, p_{a(f)})$ where $f \in \mathcal{F}$, $p_i \in \mathcal{Q}_k$ with $1 \leq i \leq a(f)$ and $\langle t'_1, \ldots, t'_m \rangle = p_1$ without loss of generality. For case (1), the subsequence of (7.30) from $u'[o \leftarrow \langle t'_1, \ldots, t'_m \rangle]$ to q can be written as:

$$\begin{aligned} u'[o \leftarrow \langle t'_1, \ldots, t'_m \rangle] &\vdash_{k^-,\bullet} \\ u'[o \leftarrow \langle t'_n \rangle] &\vdash^*_{\wedge,k^-,\bullet} \; q \end{aligned} \tag{7.31}$$

for some n $(1 \leq n \leq m)$. Let $u'' = u'[o \leftarrow t_n]$, then we have $u'' \vdash^*_{k^-,\bullet} u'[o \leftarrow \langle t'_n \rangle] \vdash^*_{\wedge,k^-,\bullet} q$ by (7.30) and (7.31). For case (2), the subsequence of (7.30) from $u'[o \leftarrow \langle t'_1, \ldots, t'_m \rangle]$ to q can be written as:

$$\begin{aligned} u'[o \leftarrow \langle t'_1, \ldots, t'_m \rangle] \;\; \vdash^*_{k^-,\bullet} \;\; & u'[o' \leftarrow f(\langle t'_1, \ldots, t'_m \rangle, \ldots, p_{a(f)})] \\ = \;\; & u'[o' \leftarrow f(p_1, \ldots, p_{a(f)})] \\ \vdash_{\wedge,k^-,\bullet} \;\; & u'[o' \leftarrow \langle t' \rangle] \\ \vdash^*_{\wedge,k^-,\bullet} \;\; & q \end{aligned} \tag{7.32}$$

where $o = o' \cdot 1$. From Step 3(iii) of **ADDTRANS**, there is a transition rule of the form $f(\langle t''_1 \rangle, \ldots, \langle t''_{a(f)} \rangle) \to \langle t' \rangle$ where $t''_i \in p_i$ for $1 \leq i \leq a(f)$. Let n' $(1 \leq n' \leq m)$ be an integer such that $t''_1 = t'_{n'}$ and assume that $u' = u'[o' \leftarrow f(t^{o'}_1, \ldots, t^{o'}_{a(f)})]$. Then, by the transition rules defined in Step 3(i) of **ADDTRANS** and (7.32), we have

$$t^{o'}_i \vdash^*_{k^-,\bullet} p_i \vdash_{k^-,\bullet} \langle t''_i \rangle \quad (2 \leq i \leq a(f)). \tag{7.33}$$

Let $u'' = u'[o \leftarrow t_{n'}]$. By (7.30) and (7.33), we have

$$\begin{aligned} u'' \;\; = \;\; & u'[o' \leftarrow f(t_{n'}, \ldots, t^{o'}_{a(f)})] \\ \vdash^*_{k^-,\bullet} \;\; & u'[o' \leftarrow f(\langle t'_{n'} \rangle, \ldots, \langle t''_{a(f)} \rangle)] \\ = \;\; & u'[o' \leftarrow f(\langle t''_1 \rangle, \ldots, \langle t''_{a(f)} \rangle)] \\ \vdash_{k^-,\bullet} \;\; & u'[o' \leftarrow \langle t' \rangle] \\ \vdash^*_{k^-,\bullet} \;\; & q. \end{aligned}$$

On the other hand, consider the rewrite sequence $u' \to^*_{\alpha,\mathcal{R}} s$. From the fact that no left-hand side has function symbols in $\wedge$ and $s, t_i \in \mathcal{T}(\mathcal{F})$ with $1 \leq i \leq m$, and from the definition of $\to_{\alpha,\mathcal{R}}$, there is an $\mathcal{F}$-term t_0 such that $u' \to^*_{\mathcal{R}} u'[o \leftarrow \wedge_m(t_0, \ldots, t_0)] \to_\alpha u'[o \leftarrow t_0] \to^*_{\alpha,\mathcal{R}} s$ where $t_0 \in \mathcal{T}(\mathcal{F})$. From this rewrite sequence, for both cases (1) and (2), we obtain $u'' \to^*_{\mathcal{R}} u'[o \leftarrow t_0] \to^*_{\alpha,\mathcal{R}} s$. Repeating the discussions above for every subterm with a function symbol in $\wedge$, we can obtain an $\mathcal{F}$-term u such that $u \to^*_{\mathcal{R}} s$ and $u \vdash^*_{k^-} q$. $\Box$

To show Lemma 5.1, it is sufficient to show that for a term $s \in \mathcal{T}(\mathcal{F})$ and a state $q \in \mathcal{Q}_0$, if $s \vdash^*_k q$, then there exists a term $u \in \mathcal{T}(\mathcal{F})$ such that $u \to^*_{\mathcal{R}} s$ and $u \vdash^*_{k^-} q$. The claim holds from Lemma 7.6.

8 Conclusion

A new class of TRS named finite path overlapping TRS (RL-FPO-TRS) is proposed. It is shown that an RL-FPO-TRS effectively preserves recognizability, and that the class properly includes known decidable classes of TRSs which effectively preserve recognizability. The result provides a positive answer for the conjecture in [7] that a right-linear semi-monadic TRS effectively preserves recognizability.

RL-FPO-TRS does not include simple EPR-TRSs such that $\mathcal{R} = \{f(x) \to f(f(x))\}$. To construct a TA $\mathcal{A}_*$ which accepts $(\to^*_{\mathcal{R}})(\mathcal{L}(\mathcal{A}))$ for a given TA $\mathcal{A}$, we might need an operation which "merges" equivalent states of a TA though such a construction of a TA may make the proofs quite complicated.

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APPENDIX: Proof of Theorem 5.7

We have observed in the proof of Lemma 5.7 in Section 5.3 that the number of layers in a state cannot varies as in Case 1(a). In this appendix, we prove that similar discussions hold for other cases and the proof completes.

Case 1(b)

In this case, the number of layers in the associated state is increased to $j+2$ or more at o' before the first rewriting move at o (see Fig. 5 1(b)). The overview of the discussion in this case is as follows: We first show that there is a rewrite rule $l' \to r'$ which corresponds to the first rewriting move, and also show that the rank of the rewrite rule is $j+1$ or more. By using the property that the number of layers is increased before the first rewriting move, a subterm of l sticks out of r', which implies that the rank of $l \to r$ must be $j+1$ or more by 3 of the definition of the sticking-out graph, a contradiction.

Let o be the position where the first rewriting move occurs and p be the state just before the rewriting move. Since the number of layers is $j+2$ or more just before the rewriting move, p has $j+2$ or more layers. By the definition of the layers, p as the set contains an element $r'\rho'$ such that $\langle r'\rho' \rangle$ has $j+2$ or more layers where $l' \to r'$ and ρ' are the rewrite rule and the $\mathcal{Q}_k$-substitution which were used to introduce $\langle r'\rho' \rangle$ at Step 4 of Procedure 5.1. The rank of the rule $l' \to r'$ must be $j+1$ or more, otherwise $\langle r'\rho' \rangle$ cannot have $j+2$ or more layers by the inductive hypothesis. In the following, we show that l/o sticks out of r'.

Consider the moves of the TA from the position o_{11} to o. Let o' be the position where the number of layers is increased first time before the first rewriting move. Since o is the first rewriting move position, all moves under o are defined by **ADDTRANS**. It follows that the function symbol of l at the position $o \cdot o''$ is the same as the function symbol of r' at o'' for every o'' such that $o \cdot o'' \prec o'$. Furthermore, since the number of layers is increased at o', o''' such that $o' = o \cdot o'''$ is a variable position of r'. This implies that l/o sticks out of r'. We have observed that the rank of $l' \to r'$ is $j+1$ or more, and thus the rank of $l \to r$ must be $j+1$ or more by 3 of the definition of the sticking-out graph, a contradiction.

Case 2(a)

In this case, the initial state p_{11} of the TA has $j+1$ or layers, and the number of layers is not changed at all in the sequence (5.1) (see Fig. 5 2(a)). That is, the last state p at the root position has the same number of layers as the initial state p_{11} has. The overview of the discussion in this case is as follows: We first show that there is a rewrite rule $l' \to r'$ such that the last state p is written as an instance of a subterm of r', and also show that the rank of the rewrite rule is j or more. By the assumption that there is no rewriting move in (5.1), the subterm of r' is shown to properly stick out of l, which implies that the rank of $l \to r$ must be $j+1$ or more by 2 of the definition of the sticking-out graph, a contradiction.

Let p be the last state which is used in the sequence (5.1) of Step 4 of Procedure 5.1 at the root position of l. Since the number of layers in the last state is $j+1$ or more, p has $j+1$ or more layers. It cannot happen that p has belonged to $\mathcal{Q}_0$ since it implies that p has only one layer. Thereby there is a rewrite rule, say $l' \to r'$ and a position o_0 such that $r'\rho'/o_0 \in p$ where ρ' is $\mathcal{Q}_k$-substitution which was introduced when $l' \to r'$ was used at Step 4 of Procedure 5.1.

The rank of the rule $l' \to r'$ must be j or more and r'/o_0 is a non-ground subterm of r' with size more than one, otherwise $\langle r'\rho'/o_0 \rangle$ cannot have $j+1$ or more layers by the inductive hypothesis. In the following, we show that r'/o_0 properly sticks out of l.

In this case, all moves at positions between the root and o_{11} are defined by **ADDTRANS**. Similarly to Case 1(a), it follows that the function symbol of l at position o'' is the same as the function symbol of $r'/o_0 \cdot o''$ for every o'' such that $o'' \prec o_{11}$. Furthermore, it can be easily shown that when the head visits position o'' ($o'' \preceq o_{11}$) of l, the state $\langle r'\rho'/o_0 \cdot o'' \rangle$ is attached to that head. Thereby, at the variable position o_{11}, the state p_{11} which contains $r'\rho'/o_0 \cdot o_{11}$ as an element was attached. This implies that r'/o_0 properly sticks out of l. We have observed that the rank of $l' \to r'$ is j or more, and thus the rank of $l \to r$ must be $j+1$ or more by 2 of the definition of the sticking-out graph, a contradiction.

Case 2(b)

In this case, the last state p of the TA at the root position has $j+2$ or more layers, and the number of layers has been increased in the sequence (5.1) (see Fig. 5 2(b)). The overview of the discussion in this case is as follows: We first show that there is a rewrite rule $l' \to r'$ such that the last state p at the root position is written as an instance of a subterm of r', and also show that the rank of the rewrite rule is $j + 1$ or more. By the assumption that the number of layers have been increased in the sequence (5.1), l sticks out of the subterm of r', which implies that the rank of $l \to r$ must be $j + 1$ or more by 4 of the definition of the sticking-out graph, a contradiction.

Let p be the last state in the sequence (5.1) of Step 4 of Procedure 5.1 at the root position of l. By assumption, p has $j + 2$ or more layers. It cannot happen that p has belonged to $\mathcal{Q}_0$ since it implies that p has only one layer. Assume $|p| = 1$. For the case $|p| > 1$, the same discussion can apply. There is a rewrite rule, say $l' \to r'$ and a position o_0 such that p has $r'\rho'/o_0$ as an element and $\text{layer}(\langle r'\rho'/o_0\rangle) \geq j + 2$ where ρ' is $\mathcal{Q}_k$-substitution which was introduced when $l' \to r'$ was used at Step 4 of Procedure 5.1. Furthermore, the rank of the rule $l' \to r'$ must be $j + 1$ or more and r'/o_0 is a non-ground subterm of r' with size more than one, otherwise $\langle r'\rho'/o_0\rangle$ cannot have $j + 2$ or more layers by the inductive hypothesis. In the following, we show that l sticks out of r'/o_0.

In this case, all moves at positions between the root and o_{11} are defined by **ADDTRANS**. Let o' be the position where the number of layers is increased first time. It follows that the function symbol of l at the position o'' is the same as the function symbol of r'/o_0 at o'' for every o'' such that $o'' \prec o'$. Furthermore, since the number of layers is increased at o', o' is a variable position of r'/o_0. This implies that l sticks out of r'/o_0. We have observed that the rank of $l' \to r'$ is $j + 1$ or more, and thus the rank of $l \to r$ must be $j + 1$ or more by 4 of the definition of the sticking-out graph, a contradiction.