Termination property of inverse finite path overlapping term rewriting system is decidable

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Abstract

We propose a new decidable subclass of term rewriting systems (TRSs) for which strongly normalizing (SN) property is decidable. The new class is called almost orthogonal inverse finite path overlapping TRSs (AO-FPO^{-1}-TRSs) and the class properly includes AO growing TRSs for which SN is decidable. Tree automata technique is used to show that SN is decidable for AO-FPO^{-1}-TRSs.

1 Introduction

Termination is a fundamental property in the theory of term rewriting systems (TRSs). However, it is undecidable whether a given TRS has the termination property or not, and this topic has been extensively studied. Those studies can be divided into two approaches. The one approach is to give sufficient conditions to guarantee termination property. Multiset ordering[4] is one of the techniques and other well-known (complete) techniques are dependency pair[1] and semantic labelling[16, 12]. The other approach is to propose (decidable) subclasses of TRSs for which the termination property is decidable. For ground TRSs[9], right-ground TRSs[5], right-linear monadic TRSs[14], termination property has been shown to be decidable. In 1999, Nagaya and Toyama proposed a class of TRSs, called growing TRSs, and showed that termination property is decidable for almost orthogonal growing TRSs[13]. Our study takes an approach similar to them. In this paper, we propose a subclass of TRSs for which termination property is decidable. This class is called almost orthogonal inverse finite path overlapping TRSs and properly includes almost orthogonal growing TRSs.

The rest of this paper is organized as follows. After defining some notions on TRSs in Section 2, we introduce almost orthogonal inverse finite path overlapping TRSs in Section 3. In Section 4, we show that the termination property is decidable for this subclass.

2 Preliminaries

We assume the reader is familiar with notions on term rewriting systems (see [2] for more details). Let $\mathcal{F}$ be a finite set of *function symbols* and $\mathcal{V}$ be an enumerable set of *variables*. The *arity* of $f \in \mathcal{F}$ is denoted by $a(f)$. $\mathcal{F}$ is called a *signature*. The set of *terms* generated by $\mathcal{F}$ and $\mathcal{V}$, defined in the usual way, is denoted by $\mathcal{T}(\mathcal{F}, \mathcal{V})$. The set of variables occurring in t is denoted by $\mathcal{V}ar(t)$. A term t is *ground* if $\mathcal{V}ar(t) = \emptyset$. The set of all ground terms is denoted by $\mathcal{T}(\mathcal{F})$. A term t is *linear* if no variable occurs more than once in t. A *substitution* σ is a mapping from $\mathcal{V}$ to $\mathcal{T}(\mathcal{F}, \mathcal{V})$, and written as $\sigma = \{x_1 \mapsto t_1, \ldots, x_n \mapsto t_n\}$ where t_i with $1 \leq i \leq n$ is a term which substitutes for the variable x_i. The term obtained by applying a substitution σ to a term t is written as $t\sigma$. $t\sigma$ is called an *instance* of t. A *position* in a term t is defined as a sequence of positive integers, and the set of all positions in a term t is denoted by $\mathcal{P}os(t)$. Let λ denote the empty sequence, which is the root position. If a position o_1 is a prefix (resp. proper prefix) of o_2, then we write $o_1 \preceq o_2$ (resp. $o_1 \prec o_2$). Two positions o_1 and o_2 are *disjoint* if neither $o_1 \preceq o_2$ nor $o_2 \preceq o_1$. A subterm of t at a position o is denoted by t/o. t/o is said to occur at *depth* $|o|$. If a term t is obtained from a term t' by replacing the subterms of t' at positions $o_1, \ldots, o_m$ ($o_i \in \mathcal{P}os(t'), o_i$ and o_j are disjoint if $i \neq j$) with terms $t_1, \ldots, t_m$, respectively, then we write $t = t'[o_i \leftarrow t_i \mid 1 \leq i \leq m]$.

A *rewrite rule* is an ordered pair of terms, written as $l \to r$. The variable restriction ($\mathcal{V}ar(r) \subseteq \mathcal{V}ar(l)$ and l is not a variable) is assumed. A *term rewriting system* (*TRS*) is a finite set of rewrite rules. For terms t, t' and a TRS $\mathcal{R}$, we write $t \to_{\mathcal{R}} t'$ if there exist a position $o \in \mathcal{P}os(t)$, a substitution σ and a rewrite rule $l \to r \in \mathcal{R}$ such that $t/o = l\sigma$ and $t' = t[o \leftarrow r\sigma]$. The relation $t \to_{\mathcal{R}} t'$ is called a *rewrite relation.* Define $\to_{\mathcal{R}}^*$ to be the reflexive and transitive closure of $\to_{\mathcal{R}}$. The subscript $\mathcal{R}$ of $\to_{\mathcal{R}}$ is omitted if $\mathcal{R}$ is clear from the context. A *redex* (*in* $\mathcal{R}$) is an instance of l for some $l \to r \in \mathcal{R}$. A *normal form* (in $\mathcal{R}$) is a term which has no redex as its subterm. Let $NF_{\mathcal{R}}$ denote the set of all ground normal forms in $\mathcal{R}$. For terms t, t' and a TRS $\mathcal{R}$, if $t = t[o \leftarrow l\sigma] \to_{\mathcal{R}} t[o \leftarrow r\sigma] = t'$ and t/o' is a normal form for any o' with $o' \in \mathcal{P}os(t)$ and $o \prec o'$, then we write $t \to_{I,\mathcal{R}} t'$ and the relation is called an *innermost rewrite step*. For a TRS $\mathcal{R}$, a term t is *strongly normalizing (SN) in* $\mathcal{R}$ if there exists no infinite rewriting sequence starting from t. A TRS $\mathcal{R}$ is *SN* if every term is SN in $\mathcal{R}$. The property SN is also called *termination*. For a TRS $\mathcal{R}$, a term t is *weakly innermost normalizing (WIN) in* $\mathcal{R}$ if there exists a normal form t' such that $t \to_{I,\mathcal{R}}^* t'$. A TRS $\mathcal{R}$ is *WIN* if every term is WIN in $\mathcal{R}$.

A rewrite rule $l \to r$ is *left-linear* (resp. *right-linear*) if l is linear (resp. r

is linear). A rewrite rule is *linear* if it is left-linear and right-linear. A TRS $\mathcal{R}$ is *left-linear* (resp. *right-linear, linear*) if every rule in $\mathcal{R}$ is left-linear (resp. right-linear, linear).

For a TRS $\mathcal{R}$, let $l_1 \to r_1$ and $l_2 \to r_2$ be (possibly the same) rewrite rules in $\mathcal{R}$ whose variables have been renamed to have no shared variables. If a non-variable subterm of l_1 at a position $o \in \mathcal{P}os(l_1)$ and l_2 are unifiable with a most general unifier σ, then the pair $r_1\sigma$ and $l_1\sigma[o \leftarrow r_2\sigma]$ is called a *critical pair of* $\mathcal{R}$ and is written as $\langle r_1\sigma, l_1\sigma[o \leftarrow r_2\sigma]\rangle$. If $l_1 \to r_1$ and $l_2 \to r_2$ are the same rewrite rule, then we do not consider the case $o = \lambda$. 0A critical pair $\langle r_1\sigma, l_1\sigma[o \leftarrow r_2\sigma]\rangle$ is an *overlay* if $o = \lambda$. A critical pair $\langle t, t'\rangle$ is *trivial*, if $t = t'$. A TRS $\mathcal{R}$ is *almost orthogonal (AO)*, if $\mathcal{R}$ is left-linear and every critical pair of $\mathcal{R}$ is a trivial overlay.

The following lemmas concerning with innermost rewrite steps can be easily understood.

Lemma 2.1 *For a term $t \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ and a TRS $\mathcal{R}$, if a rewrite step $t[o \leftarrow l\sigma] \to_{\mathcal{R}} t[o \leftarrow r\sigma]$ is innermost at a position $o \in \mathcal{P}os(t)$ with a rewrite rule $l \to r \in \mathcal{R}$ and a substitution σ, then $l\sigma \to_{\mathcal{R}} r\sigma$ is innermost.* □

Lemma 2.2 *Let $\mathcal{R}$ be an AO-TRS. For two terms $s, t \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ and a rewrite rule $l \to r \in \mathcal{R}$ if $s/o = l\sigma, t = s[o \leftarrow r\sigma]$ where $o \in \mathcal{P}os(s)$, $\sigma = \{x_i \mapsto t_i \mid 1 \le i \le n\}$, then $s \to_{I,\mathcal{R}} t$ if and only if $t_i \in NF_{\mathcal{R}}$ for $1 \le i \le n$.* □

A *tree automaton*(*TA*)[6] is defined by a 4-tuple $\mathcal{A} = (\mathcal{F}, \mathcal{Q}, \mathcal{Q}_{final}, \Delta)$ where $\mathcal{F}$ is a signature, $\mathcal{Q}$ is a finite set of state symbols, $\mathcal{Q}_{final} \subseteq \mathcal{Q}$ is a set of final states, and Δ is a finite set of transition rules of the form $f(q_1, \ldots, q_n) \to q$ where $f \in \mathcal{F}$, $a(f) = n$, and $q_1, \ldots, q_n, q \in \mathcal{Q}$ or of the form $q' \to q$ where $q, q' \in \mathcal{Q}$. The latter rule is called an ε-rule. If left-hand sides of any two distinct rules are different and there is no ε-rule, then $\mathcal{A}$ is called *deterministic*. For each $f \in \mathcal{F}$ and $q_1, \ldots, q_{a(f)} \in \mathcal{Q}$, if there exits a rule whose left-hand side is $f(q_1, \ldots, q_{a(f)})$, then $\mathcal{A}$ is *complete*. Consider the set of ground terms $\mathcal{T}(\mathcal{F} \cup \mathcal{Q})$. A *move* of a TA can be regarded as a rewrite relation on $\mathcal{T}(\mathcal{F} \cup \mathcal{Q})$ by regarding transition rules in Δ as rewrite rules on $\mathcal{T}(\mathcal{F} \cup \mathcal{Q})$. For terms t and t' in $\mathcal{T}(\mathcal{F} \cup \mathcal{Q})$, we write $t \vdash_{\mathcal{A}} t'$ if and only if $t \to_{\Delta} t'$. The reflexive and transitive closure of $\vdash_{\mathcal{A}}$ is denoted by $\vdash^*_{\mathcal{A}}$. For a TA $\mathcal{A}$ and $t \in \mathcal{T}(\mathcal{F})$, if $t \vdash^*_{\mathcal{A}} q_f$ for a final state $q_f \in \mathcal{Q}_{final}$, then we say t is *accepted* by $\mathcal{A}$. The set of ground terms over $\mathcal{F}$ accepted by $\mathcal{A}$ is denoted by $\mathcal{L}(\mathcal{A})$, and called the *(tree) language accepted by* $\mathcal{A}$. A set L of ground terms is *recognizable* if there is a TA $\mathcal{A}$ such that $L = \mathcal{L}(\mathcal{A})$. Let $\mathcal{L}_q(\mathcal{A}) = \{t \mid t \vdash^*_{\mathcal{A}} q\}$ for a state q of $\mathcal{A}$, which is called the *(tree) language* *accepted by q in $\mathcal{A}$*. For recognizable sets, some principal problems of the formal language theory are decidable[6].

Lemma 2.3 *The class of recognizable sets is effectively closed under union, intersection and complementation. For a recognizable set L, the following problems are decidable: (1) Is a given term t in L? (2) Is L empty?* □

3 FPO^{-1}-TRSs

A class of TRSs named *inverse finite path overlapping TRS* (*FPO^{-1}-TRS*) is proposed in this section. The class of FPO^{-1}-TRSs properly includes the class of growing TRSs. To define the new class, we introduce some additional notions. We say that a term s *sticks out of* t if t is not a variable and there is a variable position γ of t such that

1. for any position o with $\lambda \preceq o \prec \gamma$, we have $o \in \mathcal{P}os(s)$ and the function symbol of s at o and the function symbol of t at o are the same, and

2. $\gamma \in \mathcal{P}os(s)$ and s/γ is not a ground term.

When the position γ is of interest, we say that s sticks out of t at γ. If s sticks out of t at γ and s/γ is not a variable (i.e. s/γ is a non-ground and non-variable term), then s is said to *properly stick out of* t (at γ). For example, a term $f(g(x), a)$ sticks out of $f(g(y), b)$ at the position $1 \cdot 1$, and $f(g(g(x)), a)$ properly sticks out of $f(g(y), b)$ at the position $1 \cdot 1$. An *inverse finite path overlapping TRS* (*FPO^{-1}-TRS*) is a TRS $\mathcal{R}$ such that the following *sticking-out graph* of $\mathcal{R}$ does not have a cycle of weight one or more. The *sticking-out graph* of a TRS $\mathcal{R}$ is a directed graph $G = (V, E)$ where $V = \mathcal{R}$ (i.e. the vertices are the rewrite rules in $\mathcal{R}$) and E is defined as follows. Let v_1 and v_2 be (possibly identical) vertices which correspond to rewrite rules $l_1 \to r_1$ and $l_2 \to r_2$, respectively. Replace each variable in $\mathcal{V}ar(l_i) \setminus \mathcal{V}ar(r_i)$ with a constant not in $\mathcal{F}$, say $\diamond$, for $i = 1, 2$.

1. If l_2 properly sticks out of a subterm of r_1, then E contains an edge from v_2 to v_1 with weight one.

2. If a subterm of l_2 properly sticks out of r_1, then E contains an edge from v_2 to v_1 with weight one.

3. If a subterm of r_1 sticks out of l_2, then E contains an edge from v_2 to v_1 with weight zero.

4. If r_1 sticks out of a subterm of l_2, then E contains an edge from v_2 to v_1 with weight zero.

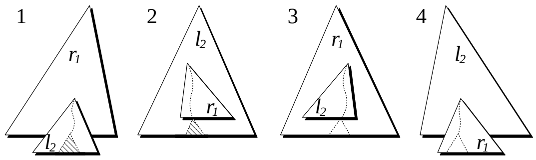

Figure 1: Sticking out relations of rewrite rules.

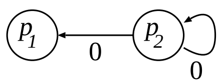

Figure 2: The sticking-out graph of $\mathcal{R}_1$.

The four cases are illustrated in Fig. 1.

Remark that the sticking-out graph is effectively constructible for a given TRS $\mathcal{R}$, and hence it is decidable whether or not a given TRS $\mathcal{R}$ is in FPO^{-1}-TRS (in $O(m^2n^2)$ time where m is the maximum size of a term in $\mathcal{R}$ and n is the number of rules in $\mathcal{R}$).

Example 3.1 Let $\mathcal{R}_1 = \{\ p_1: h(g(x)) \to f(x,x),\ p_2: f(g(x), y) \to h(f(a,y))$ $\}$. Fig. 2 shows the sticking-out graph of $\mathcal{R}_1$. The right-hand side $f(x,x)$ of p_1 sticks out of the left-hand side $f(g(x), y)$ of p_2, and hence there is an edge of weight zero from p_2 to p_1. The sticking-out graph also has an self-looping edge of weight zero at p_2 since the subterm $h(f(a,y))/1 = f(a,y)$ of the right-hand side of p_2 sticks out of its left-hand side. Since the variable x in p_2 is replaced with a constant $\diamond$, the left-hand side of p_2 does not properly stick out of the right-hand side of p_1. There is no other edge since there is no other sticking-out relation between subterms of these rewrite rules. The sticking-out graph has a cycle of weight zero, but does not have a cycle of weight one or more, and hence $\mathcal{R}_1$ is an FPO^{-1}-TRS. Let $\mathcal{R}_2 = \{g(f(g(x))) \to f(x)\}$. The subterm $f(g(x))$ of the left-hand side of the (unique) rewrite rule properly sticks out of its right-hand side, as in condition 2 of the definition of sticking-out graph. The sticking-out graph of $\mathcal{R}_2$ consists of one vertex and one cycle with weight one. Therefore, $\mathcal{R}_2$ is not an FPO^{-1}-TRS. $\square$

4 Decidability of SN for FPO^{-1}-TRSs

4.1 Relation between WIN and SN in AO-TRSs

Nagaya and Toyama[13] showed that SN is decidable for the class AO-GR-TRSs (almost orthogonal growing TRSs, the definition is presented below) in the following way.

Theorem 4.1 *[7] Let $\mathcal{R}$ be an AO-TRS. (1) $\mathcal{R}$ is SN if and only if $\mathcal{R}$ is WIN. (2) A term s is SN in $\mathcal{R}$ if and only if s is WIN in $\mathcal{R}$.* □

Definition 4.1 For a TRS $\mathcal{R}$ and a set L of terms, the *innermost $\mathcal{R}$-ancestor of L* is defined as $(\leftarrow^*_{I,\mathcal{R}})(L) = \{t \mid \exists s \in L, t \to^*_{I,\mathcal{R}} s\}$. □

By using the notion of the innermost $\mathcal{R}$-ancestor, the property WIN can be represented as follows.

Fact 4.2 *For a TRS $\mathcal{R}$, (1) $\mathcal{R}$ is WIN if and only if $(\leftarrow^*_{I,\mathcal{R}})(NF_\mathcal{R}) = \mathcal{T}(\mathcal{F})$, (2) a term s is WIN in $\mathcal{R}$ if and only if $s \in (\leftarrow^*_{I,\mathcal{R}})(NF_\mathcal{R})$.* □

By Lemma 2.3, Theorem 4.1 and Fact 4.2, we obtain the following lemma.

Lemma 4.3 *For an AO-TRS $\mathcal{R}$, if we can effectively construct a TA $\mathcal{A}_*$ such that $\mathcal{L}(\mathcal{A}_*) = (\leftarrow^*_{I,\mathcal{R}})(NF_\mathcal{R})$, then the following problems are decidable: (1) Is $\mathcal{R}$ SN? (2) For a given term s, is s SN in $\mathcal{R}$?* □

Definition 4.2 [13] A rewrite rule $l \to r$ is *growing* if every variable in $\mathcal{V}ar(l) \cap \mathcal{V}ar(r)$ does not occur in l at depth more than one. A finite set of left-linear growing rewrite rules is called a *left-linear growing TRS*(*LL-GR-TRS*). □

Neither $\mathcal{R}_1$ nor $\mathcal{R}_2$ in Example 3.1 is an LL-GR-TRS.

Theorem 4.4 *[15] GR-TRS $\subset$ FPO^{-1}-TRS.* □

Nagaya and Toyama showed that for an LL-GR-TRS $\mathcal{R}$, the set $(\leftarrow^*_{I,\mathcal{R}})(NF_\mathcal{R})$ is always recognizable and thus whether $\mathcal{R}$ is WIN or not is decidable. If $\mathcal{R}$ is also an AO-TRS, then we can decide whether $\mathcal{R}$ is SN or not by Lemma 4.3.

Theorem 4.5 *[13] For an AO-GR-TRS $\mathcal{R}$, SN is decidable.* □

In the next subsection, we show if $\mathcal{R}$ is an AO-FPO^{-1}-TRS, then a TA accepting $(\leftarrow^*_{I,\mathcal{R}})(NF_\mathcal{R})$ can be effectively constructed.

4.2 Construction of Tree Automata

For an LL-TRS $\mathcal{R}$, Comon showed a complete and deterministic TA, denoted by $\mathcal{A}_{NF_\mathcal{R}}$, which accepts the set of all ground normal forms, i.e. $\mathcal{L}(\mathcal{A}_{NF_\mathcal{R}}) = NF_\mathcal{R}$ in [3]. We start with this TA $\mathcal{A}_{NF_\mathcal{R}}$.

Procedure 4.1 This procedure takes an AO-TRS $\mathcal{R}$ as an input and outputs a TA $\mathcal{A}_*$ such that $\mathcal{L}(\mathcal{A}_*) = (\leftarrow^*_{I,\mathcal{R}})(NF_\mathcal{R})$. The algorithm does not always halt in general. We will show later that if $\mathcal{R}$ is an FPO^{-1}-TRS, then the procedure always halts. Let $\mathcal{A}_{NF_\mathcal{R}} = (\mathcal{F}, \mathcal{Q}, \mathcal{Q}_{\mathit{final}}, \Delta)$ be the complete and deterministic TA with $\mathcal{L}(\mathcal{A}_{NF_\mathcal{R}}) = NF_\mathcal{R}$[3]. We will construct TAs whose states are represented by terms in $\mathcal{T}(\mathcal{F} \cup \mathcal{Q})$ where elements in $\mathcal{Q}$ are regarded as constants. A term in $\mathcal{T}(\mathcal{F} \cup \mathcal{Q})$ is called a $\mathcal{Q}$-term, and, to avoid confusion, a $\mathcal{Q}$-term $t \in \mathcal{T}(\mathcal{F} \cup \mathcal{Q})$ is written as $\langle t \rangle$ when it is used to represent a state of TAs.

Step 1. Let $\mathcal{A}_0 = (\mathcal{F}, \mathcal{Q}_0, \mathcal{Q}^0_{\mathit{final}}, \Delta_0)$ where $\mathcal{Q}_0 = \{\langle p \rangle \mid p \in \mathcal{Q}\}$, $\mathcal{Q}^0_{\mathit{final}} = \{\langle p \rangle \mid p \in \mathcal{Q}_{\mathit{final}}\}$, and $\Delta_0 = \{f(\langle p_1 \rangle, \ldots, \langle p_{a(f)} \rangle) \to \langle p \rangle \mid f(p_1, \ldots, p_{a(f)}) \to p \in \Delta\}$. In Steps 3 to 5, $\mathcal{A}_{k+1} = (\mathcal{F}, \mathcal{Q}_{k+1}, \mathcal{Q}^0_{\mathit{final}}, \Delta_{k+1})$ $(k \geq 0)$ is constructed from $\mathcal{A}_k = (\mathcal{F}, \mathcal{Q}_k, \mathcal{Q}^0_{\mathit{final}}, \Delta_k)$ by adding states and transition rules to $\mathcal{Q}_k$ and Δ_k, respectively. We abbreviate $\vdash_{\mathcal{A}_k}$ and $\vdash^*_{\mathcal{A}_k}$ as $\vdash_k$ and $\vdash^*_k$, respectively.

Step 2. Let $k = 0$.

Step 3. Let $\mathcal{Q}_{k+1} = \mathcal{Q}_k$ and $\Delta_{k+1} = \Delta_k$.

Step 4. New states and transition rules are introduced in this step. Let $l \to r$ be a rewrite rule in $\mathcal{R}$ and let $Y = \mathcal{V}ar(l) \setminus \mathcal{V}ar(r)$. It is assumed that r has $m(\geq 0)$ variables $x_i (1 \leq i \leq m)$ and x_i occurs at positions o_{ij} in $r (1 \leq i \leq m, 1 \leq j \leq \gamma_i)$. Assume there are states $q, q_{ij} \in \mathcal{Q}_k (1 \leq i \leq m, 1 \leq j \leq \gamma_i)$ and $q_{i0} \in \mathcal{Q}^0_{\mathit{final}} (1 \leq i \leq m)$ such that

$$r[o_{ij} \leftarrow q_{ij} \mid 1 \leq i \leq m, 1 \leq j \leq \gamma_i] \vdash^*_k q \tag{4.1}$$

and $\mathbf{NO} \neq \mathbf{LUB}(\{q_{ij} \mid 0 \leq j \leq \gamma_i\})$. The function **LUB**, which is defined later, constructs a state which accepts terms accepted by every q_{ij} $(0 \leq j \leq \gamma_i)$. Then for any substitution $\rho' \colon Y \to \mathcal{Q}_{\mathit{final}}$, let $\rho = \{x_i \mapsto t_i \mid 1 \leq i \leq m\} \cup \rho'$ where $t_i = \mathbf{LUB}(\{q_{ij} \mid 0 \leq j \leq \gamma_i\})$, and do the following 1 and 2.

1. Add $\langle l\rho \rangle \to q$ to Δ_{k+1}. If $\langle l\rho \rangle \to q \in \Delta_{k+1} \setminus \Delta_k$, then the rule is called a *rewriting transition rule* of *degree* $k+1$ and if a move of the TA is caused by this rule, then the move is called a *rewriting move* of *degree* $k+1$.

2. Execute **ADDTRANS**($\langle l\rho \rangle$). In **ADDTRANS**($\langle l\rho \rangle$), new transition rules are defined so that $l\rho \vdash^*_{k+1} \langle l\rho \rangle$.

Simultaneously execute this Step 4 for every rewrite rule and every tuple of states that satisfy the condition (4.1) and every substitution $\rho' \colon Y \to \mathcal{Q}^0_{final}$.

Step 5. If $\Delta_{k+1} = \Delta_k$ then output $\mathcal{A}_k$ as $\mathcal{A}_*$ and halt. Otherwise, let $k = k+1$ and go to Step 3. □

Procedure 4.2 [ADDTRANS] This procedure takes a state $\langle t \rangle$ as an input. If $\langle t \rangle$ already exists in $\mathcal{Q}_k$, then the procedure performs nothing. Otherwise, the procedure adds $\langle t \rangle$ to $\mathcal{Q}_k$ and defines new transition rules as follows.

Case 1. If $t = c$ with c a constant, then define $c \to \langle c \rangle$ as a transition rule.

Case 2. If $t = f(t_1, \ldots, t_{a(f)})$ with $f \in \mathcal{F}$, then define $f(\langle t_1 \rangle, \ldots, \langle t_{a(f)} \rangle) \to \langle t \rangle$ as a transition rule and execute **ADDTRANS**($\langle t_i \rangle$) for $1 \le i \le a(f)$. □

In the following, we will use $t, t', t_1, t_2, \ldots$ to denote $\mathcal{Q}$-terms in $\mathcal{T}(\mathcal{F} \cup \mathcal{Q})$, $s, s', u, u', u_1, u_2, \ldots$ to denote ground terms in $\mathcal{T}(\mathcal{F})$, $f, g, \ldots$ to denote function symbols. Also $q, q_1, q_2, \ldots$ are states in $\mathcal{Q}_k$ for some $k \ge 0$ and $p, p_1, p_2 \ldots$ are states in $\mathcal{Q}$. If we write $f(t_1, \ldots, t_{a(f)})$, then we implicitly include the case when $a(f) = 0$.

In order to define the function **LUB**, we introduce a partial order on $\mathcal{T}(\mathcal{F} \cup \mathcal{Q})$. For a $\mathcal{Q}$-term t, let $t\langle\rangle$ denote the term obtained from t by replacing every $p \in \mathcal{Q}$ in t with $\langle p \rangle$. For example, if $t = f(g(p_1), p_2)$ where $p_1, p_2 \in \mathcal{Q}$, then $t\langle\rangle = f(g(\langle p_1 \rangle), \langle p_2 \rangle)$. Note that if $s \in \mathcal{T}(\mathcal{F})$ then $s\langle\rangle = s$. The relation $\le$ on $\mathcal{T}(\mathcal{F} \cup \mathcal{Q})$ is defined as follows: (1) for $p \in \mathcal{Q}$ and $t' \in \mathcal{T}(\mathcal{F} \cup \mathcal{Q})$, if $t'\langle\rangle \vdash^*_0 \langle p \rangle$, then $p \le t'$ and (2) for $f(t_1, \ldots, t_{a(f)}), f(t'_1, \ldots, t'_{a(f)}) \in \mathcal{T}(\mathcal{F} \cup \mathcal{Q})$, if $t_i \le t'_i$ $(1 \le i \le a(f))$ then $f(t_1, \ldots, t_{a(f)}) \le f(t'_1, \ldots, t'_{a(f)})$.

Note that if $p \in \mathcal{Q}$, then $p \le p$ by (1). If $p, p' \in \mathcal{Q}$ and $p \ne p'$ then $p\langle\rangle = \langle p \rangle \not\vdash^*_0 \langle p' \rangle$ (since A_0 is deterministic) and hence $p \not\le p'$ by (1).

For two $\mathcal{Q}$-terms t and t' if there is the least upper bound of t and t' on $\le$, then it is denoted by $t \sqcup t'$. It is easily shown that $t \sqcup t'$ is represented as follows:

$$
\begin{array}{l}
t \sqcup t' = \\
t \quad \text{if} \quad t = t' \in \mathcal{Q} \\
t \quad \text{if} \quad t \in \mathcal{T}(\mathcal{F} \cup \mathcal{Q}) \setminus \mathcal{Q}, t' \in \mathcal{Q}, t\langle\rangle \vdash_0^* \langle t' \rangle \\
t' \quad \text{if} \quad t \in \mathcal{Q}, t' \in \mathcal{T}(\mathcal{F} \cup \mathcal{Q}) \setminus \mathcal{Q}, t'\langle\rangle \vdash_0^* \langle t \rangle \\
f(t_1 \sqcup t'_1, \ldots, t_{a(f)} \sqcup t'_{a(f)}) \\
\quad \text{if} \quad t = f(t_1, \ldots, t_{a(f)}) \in \mathcal{T}(\mathcal{F} \cup \mathcal{Q}) \setminus \mathcal{Q}, \\
\quad\quad t' = f(t'_1, \ldots, t'_{a(f)}) \in \mathcal{T}(\mathcal{F} \cup \mathcal{Q}) \setminus \mathcal{Q}, \\
\quad\quad t_i \sqcup t'_i \text{ defined } (1 \leq i \leq a(f)) \\
\text{undefined otherwise.}
\end{array}
\tag{4.2}
$$

For $k \geq 0$, let $\mathcal{A}_{k^-}$ be the TA obtained from $\mathcal{A}_k$ by removing every rewriting transition rule.

Function 4.1 [LUB] This function takes a set of states $\{\langle t_1 \rangle, \ldots, \langle t_n \rangle\}$ as an input and returns a $\mathcal{Q}$-term $t = t_1 \sqcup \cdots \sqcup t_n$ if it is defined. Also the function adds new transition rules and states so that $\mathcal{L}_{\langle t_1 \rangle}(\mathcal{A}_{k^-}) \cap \cdots \cap \mathcal{L}_{\langle t_n \rangle}(\mathcal{A}_{k^-}) = \mathcal{L}_{\langle t \rangle}(\mathcal{A}_{(k+1)^-})$.

Step 1. Decide whether $t_1 \sqcup \cdots \sqcup t_n$ is defined by using (4.2). If defined then let $t = t_1 \sqcup \cdots \sqcup t_n$ and go to Step 2. Otherwise, return **NO**.

Step 2. Execute **ADDTRANS**($\langle t \rangle$) and return t. □

Example 4.1 We apply Procedure 4.1 to the AO-FPO^{-1}-TRS $\mathcal{R}_1$ in Example 3.1. First, we construct the deterministic and complete TA $\mathcal{A}_0$ accepting $NF_{\mathcal{R}_1}$ as $\mathcal{A}_0 = (\mathcal{F}, \mathcal{Q}_0, \mathcal{Q}^0_{final}, \Delta_0)$ where $\mathcal{Q}_0 = \{\langle p_r \rangle, \langle p_0 \rangle, \langle p_1 \rangle\}$, $\mathcal{Q}^0_{final} = \{\langle p_0 \rangle, \langle p_1 \rangle\}$ and $\Delta_0 = \{$

$$
\begin{array}{rclrcl}
a & \to & \langle p_0 \rangle, & g(\langle p_0 \rangle) & \to & \langle p_1 \rangle \\
g(\langle p_1 \rangle) & \to & \langle p_1 \rangle, & g(\langle p_r \rangle) & \to & \langle p_r \rangle \\
h(\langle p_0 \rangle) & \to & \langle p_0 \rangle, & h(\langle p_1 \rangle) & \to & \langle p_r \rangle \\
h(\langle p_r \rangle) & \to & \langle p_r \rangle, & f(\langle p_r \rangle, \langle p_r \rangle) & \to & \langle p_r \rangle \\
f(\langle p_0 \rangle, \langle p_r \rangle) & \to & \langle p_r \rangle, & f(\langle p_1 \rangle, \langle p_r \rangle) & \to & \langle p_r \rangle \\
f(\langle p_r \rangle, \langle p_0 \rangle) & \to & \langle p_r \rangle, & f(\langle p_r \rangle, \langle p_1 \rangle) & \to & \langle p_r \rangle \\
f(\langle p_1 \rangle, \langle p_0 \rangle) & \to & \langle p_r \rangle, & f(\langle p_1 \rangle, \langle p_1 \rangle) & \to & \langle p_r \rangle \\
f(\langle p_0 \rangle, \langle p_0 \rangle) & \to & \langle p_0 \rangle, & f(\langle p_0 \rangle, \langle p_1 \rangle) & \to & \langle p_0 \rangle\}.
\end{array}
$$

Consider the rewrite rule $h(g(x)) \to f(x, x)$ in Step 4 for $\mathcal{A}_0$ ($k = 0$). Since a move $f(\langle p_0 \rangle, \langle p_0 \rangle) \vdash_0 \langle p_0 \rangle$ is possible and **LUB**($\{\langle p_0 \rangle, \langle p_0 \rangle\}$) $= p_0$, the substitution ρ in Step 4 is $\rho = \{x \mapsto p_0\}$ and new transition rules $\langle h(g(p_0)) \rangle \to \langle p_0 \rangle$, $h(\langle g(p_0) \rangle) \to \langle h(g(p_0)) \rangle$, $g(\langle p_0 \rangle) \to \langle g(p_0) \rangle$ are added to Δ_1. The last two rules are added in **ADDTRANS**($\langle h(g(p_0)) \rangle$).

Next, consider the rewrite rule $f(g(x), y) \to h(f(a, y))$ in Step 4 for $\mathcal{A}_0$. In this case, we need to consider two substitutions $\{x \mapsto p_0\}$ and $\{x \mapsto p_1\}$ as ρ'. Since $h(f(a, \langle p_0 \rangle)) \vdash_0^* \langle p_0 \rangle$ is possible, $\langle f(g(p_0), p_0) \rangle \to \langle p_0 \rangle$, $\langle f(g(p_1), p_0) \rangle \to \langle p_0 \rangle$ are added to Δ_1 and **ADDTRANS**$(\langle f(g(p_0), p_0) \rangle)$ and **ADDTRANS**$(\langle f(g(p_1), p_0) \rangle)$ are executed. We also have $h(f(a, \langle p_1 \rangle)) \vdash_0^* \langle p_0 \rangle$ and hence we define $\langle f(g(p_0), p_1) \rangle \to \langle p_0 \rangle$, $\langle f(g(p_1), p_1) \rangle \to \langle p_0 \rangle$ as new transition rules in Δ_1 and both **ADDTRANS** $(\langle f(g(p_0), p_1) \rangle)$ and **ADDTRANS** $(\langle f(g(p_1), p_1) \rangle)$ are executed. Again consider the rewrite rule $h(g(x)) \to f(x, x)$ in Step 4 for $\mathcal{A}_1$. Since a move $f(\langle g(p_0) \rangle, \langle p_1 \rangle) \vdash_1^* \langle p_0 \rangle$ is possible and $g(p_0) \sqcup p_1 = g(p_0)$, a new transition rule $\langle h(g(g(p_0))) \rangle \to \langle p_0 \rangle$ is added to Δ_2 and **ADDTRANS**$(\langle h(g(g(p_0))) \rangle)$ is executed. We can easily verify that $\mathcal{A}_2$ accepts $\mathcal{T}(\mathcal{F})$. □

Lemma 4.6 *For a state $q \in \mathcal{Q}_k$ $(k \geq 0)$, $\mathcal{L}_q(\mathcal{A}_{k^-}) = \mathcal{L}_q(\mathcal{A}_{k'^-})$ for any $k' \geq k$. (Especially, for a state $q \in \mathcal{Q}_0$, $\mathcal{L}_q(\mathcal{A}_0) = \mathcal{L}_q(\mathcal{A}_{k^-})$ for any $k \geq 0$.)*

Proof. $\mathcal{L}_q(\mathcal{A}_{k^-}) \subseteq \mathcal{L}_q(\mathcal{A}_{k'^-})$ is obvious since the sets of states and rules are enlarged monotonically. Assume $\exists t \in \mathcal{L}_q(\mathcal{A}_{k'^-}) \setminus \mathcal{L}_q(\mathcal{A}_{k^-})$. Then there exists an outermost position $\exists o \in \mathcal{P}os(t)$ where a rule $f(q'_1, \ldots, q'_{a(f)}) \to q'$ in $\Delta_{k'} \setminus \Delta_k$ is used. Since o is an outermost among such positions, $q' \in \mathcal{Q}_k$. However, to define a new transition rule whose right-hand side is q', **ADDTRANS**(q') must be executed. Since q' has been already included in $\mathcal{Q}_k$, **ADDTRANS** does not introduce any rules, a contradiction. □

Lemma 4.7 *Assume $\langle f(t_1, \ldots, t_{a(f)}) \rangle \in \mathcal{Q}_k$.*

1. *For $u \in \mathcal{T}(\mathcal{F})$, if $u \vdash_{k^-}^* \langle f(t_1, \ldots, t_{a(f)}) \rangle$, then $u = f(u_1, \ldots, u_{a(f)}) \vdash_{k^-}^* f(\langle t_1 \rangle, \ldots, \langle t_{a(f)} \rangle) \vdash_{k^-} \langle f(t_1, \ldots, t_{a(f)}) \rangle$ for some $u_i \in \mathcal{T}(\mathcal{F})$ $(1 \leq i \leq a(f))$.*

2. *$\mathcal{L}_{\langle f(t_1, \ldots, t_{a(f)}) \rangle}(\mathcal{A}_{k^-}) = \{f(u_1, \ldots, u_{a(f)}) \mid u_i \in \mathcal{L}_{\langle t_i \rangle}(\mathcal{A}_{k^-}), 1 \leq i \leq a(f)\}$. (Especially, $\mathcal{L}_{\langle c \rangle}(\mathcal{A}_{k^-}) = \{c\}$.)*

Proof. The non-rewriting transition rule defined in Procedures 4.1 and 4.2 whose right-hand side is $\langle f(t_1, \ldots, t_{a(f)}) \rangle$ must be $f(\langle t_1 \rangle, \ldots, \langle t_{a(f)} \rangle) \to \langle f(t_1, \ldots, t_{a(f)}) \rangle$. Those transition rules are defined in **ADDTRANS**. □

Lemma 4.8 *For a state $\langle t \rangle \in \mathcal{Q}_k$, $t \leq u$ for any term u in $\mathcal{L}_{\langle t \rangle}(\mathcal{A}_{k^-})$.*

Proof. For a state $\langle t \rangle \in \mathcal{Q}_0$, the lemma holds obviously since $u = u\langle \rangle \vdash_0^* \langle t \rangle$ by Lemma 4.6 and hence $t \leq u$ by (1) in the definition of $\leq$. For a state $\langle t \rangle \in \mathcal{Q}_k \setminus \mathcal{Q}_0$, **ADDTRANS**$(\langle t \rangle)$ must have been executed. We show the lemma holds for $\langle t \rangle$ by structural induction on the term t. For $t =$

$f(t_1,\ldots,t_{a(f)})$, $\mathcal{L}_{\langle t\rangle}(\mathcal{A}_{k^-}) = \{f(u_1,\ldots,u_{a(f)}) \mid u_i \in \mathcal{L}_{\langle t_i\rangle}(\mathcal{A}_{k^-}), 1 \le i \le a(f)\}$ by Lemma 4.7(2). By the inductive hypothesis, for any $u_i \in \mathcal{L}_{\langle t_i\rangle}(\mathcal{A}_{k^-})$ $t_i \le u_i$ holds. Thus, for any term $u \in \mathcal{L}_{\langle t\rangle}(\mathcal{A}_{k^-})$, $t \le u$ holds by (2) in the definition of $\le$. □

Lemma 4.9 *For a term $u \in \mathcal{T}(\mathcal{F})$ and a state $\langle l\rho\rangle$ where l is a linear term in $\mathcal{T}(\mathcal{F},\mathcal{V})$ and ρ is a substitution $\rho = \{x_i \mapsto t_i \mid 1 \le i \le n, t_i \in \mathcal{T}(\mathcal{F}\cup\mathcal{Q})\}$, if $u \vdash^*_{k^-} \langle l\rho\rangle$, then there is a substitution $\sigma\colon \mathcal{V}ar(l) \to \mathcal{T}(\mathcal{F})$ such that $u = l\sigma$ and the sequence $u \vdash^*_{k^-} \langle l\rho\rangle$ can be written as $u \vdash^*_{k^-} l\rho' \vdash^*_{k^-} \langle l\rho\rangle$ where $\rho' = \{x_i \mapsto \langle t_i\rangle \mid 1 \le i \le n\}$.*

Proof. We need to show that (1) there is σ with $u = l\sigma$, (2) $u \vdash^*_{k^-} l\rho'$ and (3) $l\rho' \vdash^*_{k^-} \langle l\rho\rangle$. For (1), assume that x_i occurs at o_i in l for $1 \le i \le n$. Using Lemma 4.8, $u \vdash^*_{k^-} \langle l\rho\rangle$ implies $l\rho \le u$, and therefore $o_i \in \mathcal{P}os(u)$. Define $\sigma = \{x_i \mapsto u/o_i \mid 1 \le i \le n\}$, then $u = l\sigma$. (3) is rather obvious from the construction of transition rules in **ADDTRANS**, and hence (2) is shown hereafter. For the proof, it suffices to show that $u/o \vdash^*_{k^-} \langle l\rho/o\rangle$ for all $o \in \mathcal{P}os(l)$, since this will imply $u/o_i \vdash^*_{k^-} \langle l\rho/o_i\rangle = \langle t_i\rangle$ and therefore $u \vdash^*_{k^-} l\rho'$. The proof is by induction on the length of o. The claim holds for the case $|o| = 0$ by the assumption $u \vdash^*_{k^-} \langle l\rho\rangle$. Assume that $l\rho/o = f(t_1,\ldots,t_{a(f)})$ and

$$u/o \vdash^*_{k^-} \langle l\rho/o\rangle \tag{4.3}$$

as an inductive hypothesis. By Lemma 4.7(1), (4.3) can be written as

$$u/o \vdash^*_{k^-} f(\langle t_1\rangle,\ldots,\langle t_{a(f)}\rangle) \vdash_{k^-} \langle l\rho/o\rangle.$$

Hence, $u/o\cdot i \vdash^*_{k^-} \langle t_i\rangle = \langle l\rho/o\cdot i\rangle$ for $1 \le i \le a(f)$. □

For example, assume that $u = f(g(c),h(a))$, $l = f(x,h(y))$, $\rho = \{x \mapsto g(p_1), y \mapsto p_2\}$ and $u \vdash^*_{k^-} \langle f(g(p_1),h(p_2))\rangle (= \langle l\rho\rangle)$. Lemma 4.9 states that

$$\begin{aligned} u &= f(g(c),h(a)) \\ &\vdash^*_{k^-} f(\langle g(p_1)\rangle, h(\langle p_2\rangle)) \vdash^*_{k^-} \langle f(g(p_1),h(p_2))\rangle. \end{aligned}$$

Lemma 4.9 implies the following corollary as a special case.

Corollary 4.10 *For $u \in \mathcal{T}(\mathcal{F})$ and $\langle t\rangle \in \mathcal{Q}_k$, if $u \vdash^*_{k^-} \langle t\rangle$ then $u \vdash^*_{k^-} t\langle\rangle \vdash^*_{k^-} \langle t\rangle$.* □

The following two lemmas show the correctness of the function **LUB**.

Lemma 4.11 *For states $\langle t_1\rangle,\ldots,\langle t_n\rangle$ in $\mathcal{Q}_k$, if $\mathcal{L}_{\langle t_1\rangle}(\mathcal{A}_{k^-})\cap\cdots\cap\mathcal{L}_{\langle t_n\rangle}(\mathcal{A}_{k^-}) \ne \emptyset$, then $t_1 \sqcup \cdots \sqcup t_n$ is defined.*

Proof. For simplicity, we prove the lemma only for $n = 2$. (An inductive argument can apply to the case when $n \geq 3$.) Assume

$$\mathcal{L}_{\langle t\rangle}(\mathcal{A}_{k^-}) \cap \mathcal{L}_{\langle t'\rangle}(\mathcal{A}_{k^-}) \neq \emptyset. \tag{4.4}$$

We prove the lemma by the structural induction on t. There are four cases to consider.

- If $t, t' \in \mathcal{Q}$ then $t = t'$ by (4.4), Lemma 4.6 and the fact that A_0 is deterministic. Hence, $t \sqcup t'$ is defined as t by (4.2).

- Assume $t \in \mathcal{T}(\mathcal{F} \cup \mathcal{Q}) \setminus \mathcal{Q}$ and $t' \in \mathcal{Q}$. By (4.4), there exists a term $u \in \mathcal{L}_{\langle t\rangle}(\mathcal{A}_{k^-}) \cap \mathcal{L}_{\langle t'\rangle}(\mathcal{A}_{k^-})$. Thus $u \vdash^*_{k^-} \langle t\rangle$ and $u \vdash^*_{k^-} \langle t'\rangle$. By Corollary 4.10,
$$u \vdash^*_{k^-} t\langle\rangle \vdash^*_{k^-} \langle t\rangle. \tag{4.5}$$
Since $\mathcal{A}_0$ is complete, there exists a state $\langle p\rangle \in \mathcal{Q}_0$ such that $t\langle\rangle \vdash^*_0 \langle p\rangle$, which implies $u \vdash^*_{k^-} \langle p\rangle$ by (4.5). Since $u \vdash^*_{k^-} \langle t'\rangle$ and $u \vdash^*_{k^-} \langle p\rangle$, we see that $u \vdash^*_0 \langle t'\rangle$ and $u \vdash^*_0 \langle p\rangle$ by Lemma 4.6, which implies $t' = p$ by the determinicity of $\mathcal{A}_0$. Hence, $t\langle\rangle \vdash^*_0 \langle t'\rangle$ and $t \sqcup t'$ is defined as t by (4.2).

- For the case when $t \in \mathcal{Q}$ and $t' \in \mathcal{T}(\mathcal{F} \cup \mathcal{Q}) \setminus \mathcal{Q}$, the claim can be proved in a similar way.

- Assume that $t = f(t_1, \ldots, t_{a(f)})$ and $t' = f(t'_1, \ldots, t'_{a(f)})$. It follows from Lemma 4.7(2) that (4.4) implies $\mathcal{L}_{\langle t_i\rangle}(\mathcal{A}_{k^-}) \cap \mathcal{L}_{\langle t'_i\rangle}(\mathcal{A}_{k^-}) \neq \emptyset$ $(1 \leq i \leq a(f))$. By the inductive hypothesis, $t_i \sqcup t'_i$ is defined for $1 \leq i \leq a(f)$. Hence, $t \sqcup t'$ is defined by (4.2).

□ Although Lemma 4.11 and the following lemma have duality, we divide them because of some technical reasons.

Lemma 4.12 *For states $\langle t_1\rangle, \ldots, \langle t_n\rangle$ in $\mathcal{Q}_k$, if $t = t_1 \sqcup \cdots \sqcup t_n$ is defined and $\langle t\rangle \in \mathcal{Q}_k$, then $\mathcal{L}_{\langle t\rangle}(\mathcal{A}_{k^-}) = \mathcal{L}_{\langle t_1\rangle}(\mathcal{A}_{k^-}) \cap \cdots \cap \mathcal{L}_{\langle t_n\rangle}(\mathcal{A}_{k^-})$.*

Proof. Again, we prove the lemma only for $n = 2$ by the structural induction. Assume $t \sqcup t'$ is defined. We perform case analysis according to (4.2).

- If $t = t' \in \mathcal{Q}$, then clearly the lemma holds.

- If $t \in \mathcal{T}(\mathcal{F} \cup \mathcal{Q}) \setminus \mathcal{Q}$, $t' \in \mathcal{Q}$, then $t \sqcup t'$ must be t and $t\langle\rangle \vdash^*_0 \langle t'\rangle$. For any term $u \in \mathcal{L}_{\langle t\rangle}(\mathcal{A}_{k^-})$ (i.e. $u \vdash^*_{k^-} \langle t\rangle$), $u \vdash^*_{k^-} t\langle\rangle \vdash^*_{k^-} \langle t\rangle$ by Corollary 4.10 and thus $u \vdash^*_{k^-} \langle t'\rangle$ by the assumption. Hence, $\mathcal{L}_{\langle t\rangle}(\mathcal{A}_{k^-}) \subseteq \mathcal{L}_{\langle t'\rangle}(\mathcal{A}_{k^-})$ and the lemma holds.

- The case when $t \in \mathcal{Q}$, $t \in \mathcal{T}(\mathcal{F} \cup \mathcal{Q}) \setminus \mathcal{Q}$ and $t'\langle\rangle \vdash_0^* \langle t \rangle$ is similar.
- Assume $t = f(t_1, \ldots, t_{a(f)}), t' = f(t'_1, \ldots, t'_{a(f)})$ and $t_i \sqcup t'_i$ are defined for $1 \leq i \leq a(f)$. Then, $t \sqcup t' = f(t_1 \sqcup t'_1, \ldots, t_{a(f)} \sqcup t'_{a(f)})$. By the inductive hypothesis,
$$\mathcal{L}_{\langle t_i \sqcup t'_i \rangle}(\mathcal{A}_{k^-}) = \mathcal{L}_{\langle t_i \rangle}(\mathcal{A}_{k^-}) \cap \mathcal{L}_{\langle t'_i \rangle}(\mathcal{A}_{k^-}). \tag{4.6}$$
for $1 \leq i \leq a(f)$. Thus,
$$\begin{array}{ll} & \mathcal{L}_{\langle t \sqcup t' \rangle}(\mathcal{A}_{k^-}) \\ = & \mathcal{L}_{\langle f(t_1 \sqcup t'_1, \ldots, t_{a(f)} \sqcup t'_{a(f)}) \rangle}(\mathcal{A}_{k^-}) \\ = & \{f(u_1, \ldots, u_{a(f)}) \mid u_i \in \mathcal{L}_{\langle t_i \sqcup t'_i \rangle}(\mathcal{A}_{k^-})\} \\ & \quad \text{(by Lemma 4.7(2))} \\ = & \{f(u_1, \ldots, u_{a(f)}) \mid \\ & \quad u_i \in \mathcal{L}_{\langle t_i \rangle}(\mathcal{A}_{k^-}) \cap \mathcal{L}_{\langle t'_i \rangle}(\mathcal{A}_{k^-})\} \\ & \quad \text{(by (4.6))} \\ = & \{f(u_1, \ldots, u_{a(f)}) \mid u_i \in \mathcal{L}_{\langle t_i \rangle}(\mathcal{A}_{k^-})\} \\ & \quad \cap \{f(u_1, \ldots, u_{a(f)}) \mid u_i \in \mathcal{L}_{\langle t_i \rangle}(\mathcal{A}_{k^-})\} \\ = & \mathcal{L}_{\langle f(t_1, \ldots, t_{a(f)}) \rangle}(\mathcal{A}_{k^-}) \\ & \quad \cap \mathcal{L}_{\langle f(t'_1, \ldots, t'_{a(f)}) \rangle}(\mathcal{A}_{k^-}) \\ & \quad \text{(by Lemma 4.7(2)).} \end{array}$$

□

4.3 Soundness

Lemma 4.13 *For a term $s \in \mathcal{T}(\mathcal{F})$ and states $q, q_0 \in \mathcal{Q}_k$, if $s \vdash_{k^-}^* q \vdash_k q_0$ where the move $q \vdash_k q_0$ is a rewriting move, then there is a term $s' \in \mathcal{T}(\mathcal{F})$ such that $s \rightarrow_{I,\mathcal{R}} s'$ and $s' \vdash_{k-1}^* q_0$.*

Proof. Assume that the move $q \vdash_k q_0$ is caused by the rewriting transition rule $q \rightarrow q_0$ of degree d $(\leq k)$ and $q \rightarrow q_0$ is defined for a rewrite rule $l \rightarrow r$ in Step 4 of Procedure 4.1. Therefore q can be written as $q = \langle l\rho \rangle \in \mathcal{Q}_d$ where $\rho = \{x_i \mapsto t_i \mid 1 \leq i \leq n\}$. Also assume that r has m variables $x_1, \ldots, x_m$ and the variable x_i occurs at o_{ij} in r $(1 \leq i \leq m, 1 \leq j \leq \gamma_i)$ and at o_i in l $(1 \leq i \leq n)$. By applying Lemmas 4.6 and 4.9 to $s \vdash_{k^-}^* \langle l\rho \rangle$, there is a substitution σ such that $s = l\sigma$ and
$$x_i\sigma \vdash_{d^-}^* \langle t_i \rangle \quad (1 \leq i \leq n). \tag{4.7}$$
By Step 4 of Procedure 4.1, there are states $q_{ij} \in \mathcal{Q}_{d-1}$ and $q_{i0} \in \mathcal{Q}_{final}^0$ such that
$$r[o_{ij} \leftarrow q_{ij} \mid 1 \leq i \leq m, 1 \leq j \leq \gamma_i] \vdash_{d-1}^* q_0, \tag{4.8}$$

and $t_i = \mathbf{LUB}(\{q_{ij} \mid 0 \leq j \leq \gamma_i\})$. By (4.7) and Lemmas 4.6, 4.12, the following moves are possible:

$$x_i\sigma \vdash^*_{(d-1)^-} q_{ij} \quad (1 \leq i \leq m, 0 \leq j \leq \gamma_i). \tag{4.9}$$

Since $q_{i0} \in \mathcal{Q}^0_{\mathit{final}}$ and also $x_i\sigma \vdash^*_0 q_{i0}$ by Lemma 4.6 and (4.9),

$$x_i\sigma \in NF_{\mathcal{R}} \tag{4.10}$$

for $1 \leq i \leq n$ (For the case $x_i \in \mathcal{V}ar(l) \setminus \mathcal{V}ar(r)$, (4.10) trivially holds since $\langle t_i \rangle \in \mathcal{Q}^0_{\mathit{final}}$ by the definition of ρ in Step 4 of Procedure 4.1). Let $s' = r\sigma$, then $s \to_{I,\mathcal{R}} s'$ from (4.10) and Lemma 2.2. On the other hand, by (4.8), (4.9) and Lemma 4.6, $r\sigma \vdash^*_{k-1} q_0$ and the lemma holds. □

The next lemma shows the soundness of Procedure 4.1.

Lemma 4.14 *For a term s and a state $q \in \mathcal{Q}_k$, if $s \vdash^*_k q$, then there is a term s' such that $s \to^*_{I,\mathcal{R}} s'$ and $s' \vdash^*_{k^-} q$.*

Proof. The proof is shown by induction on the highest degree d of rewriting moves in $s \vdash^*_k q$. For the base case ($d = 0$, which means $s \vdash^*_{k^-} q$), let $s' = s$ and the lemma holds. Assume the lemma holds for the highest degree less than d and consider the case with d. The inductive part is shown by induction on the number m (≥ 1) of rewriting moves of degree d. The sequence that has m rewriting moves of degree d can be written as

$$s \vdash^*_k s[o \leftarrow q'] \vdash_k s[o \leftarrow q_0] \vdash^*_k q \tag{4.11}$$

where $o \in \mathcal{P}os(s)$, $q', q \in \mathcal{Q}_k$ and the move $s[o \leftarrow q'] \vdash_k s[o \leftarrow q_0]$ is the first rewriting move of degree d in the sequence (that is, o is one of the innermost position of the rewriting move of degree d). From the definition of TAs, we have

$$s/o \vdash^*_k q' \vdash_k q_0 \tag{4.12}$$

and by the inductive hypothesis for d, there is a term u such that

$$s/o \to^*_{I,\mathcal{R}} u \tag{4.13}$$

and

$$u \vdash^*_{k^-} q'. \tag{4.14}$$

From (4.12) and (4.14), we have

$$u \vdash^*_{k^-} q' \vdash_k q_0. \tag{4.15}$$

Applying Lemma 4.13 to (4.15), we can see that there is a term v such that

$$u \to_{I,\mathcal{R}} v \tag{4.16}$$

and

$$v \vdash^*_{k-1} q_0. \tag{4.17}$$

From (4.11) and (4.17), we obtain $s[o \leftarrow v] \vdash^*_{k-1} s[o \leftarrow q_0] \vdash^*_k q$ which have only $m-1$ rewriting moves of degree d. By the inductive hypothesis for m (if $m-1 \geq 1$) or by the inductive hypothesis for d (if $m-1=0$), there is a term s' such that

$$s[o \leftarrow v] \to^*_{I,\mathcal{R}} s' \tag{4.18}$$

and $s' \vdash^*_{k^-} q$. From (4.13),(4.16) and (4.18), we have $s[o \leftarrow s/o] = s \to^*_{I,\mathcal{R}} s[o \leftarrow u] \to_{I,\mathcal{R}} s[o \leftarrow v] \to^*_{I,\mathcal{R}} s'$ and the lemma holds. □

4.4 Completeness

Lemma 4.15 *For a rewrite rule $l \to r \in \mathcal{R}$, a substitution σ and a state $q \in \mathcal{Q}_k$, if $l\sigma \to_{I,\mathcal{R}} r\sigma$ and $r\sigma \vdash^*_k q$, then $l\sigma \vdash^*_{k+1} q$ holds.*

Proof. Assume l has variables $x_1, \ldots, x_n$ and each x_i occurs at o_{ij} in r for $1 \leq i \leq m$ and $1 \leq i \leq \gamma_i$. From the assumption $l\sigma \to_{I,\mathcal{R}} r\sigma$,

$$x_i\sigma \in NF_{\mathcal{R}} \quad (1 \leq i \leq n). \tag{4.19}$$

The sequence $r\sigma \vdash^*_k q$ can be written as

$$\begin{aligned} r\sigma &\vdash^*_k r[o_{ij} \leftarrow q_{ij} \mid 1 \leq i \leq m, 1 \leq j \leq \gamma_i] \\ &\vdash^*_k q. \end{aligned} \tag{4.20}$$

From (4.20), we obtain

$$x_i\sigma \vdash^*_k q_{ij}. \tag{4.21}$$

By applying Lemma 4.14 to (4.21), we can see that there are terms u_{ij} with $1 \leq i \leq m$ and $1 \leq j \leq \gamma_i$ such that $u_{ij} \vdash^*_{k^-} q_{ij}$ and $x_i\sigma \to^*_{I,\mathcal{R}} u_{ij}$. Since $x_i\sigma$ is in normal form, $x_i\sigma = u_{ij}$ and hence

$$x_i\sigma \vdash^*_{k^-} q_{ij}. \tag{4.22}$$

By (4.19), there are states $q_{i0} \in \mathcal{Q}^0_{\mathit{final}}$ such that

$$x_i\sigma \vdash^*_0 q_{i0} \quad (1 \leq i \leq n). \tag{4.23}$$

By (4.22), (4.23) and Lemma 4.6, $\mathcal{L}_{q_{i1}}(\mathcal{A}_{k^-}) \cap \cdots \cap \mathcal{L}_{q_i\gamma_i}(\mathcal{A}_{k^-}) \cap \mathcal{L}_{q_{i0}}(\mathcal{A}_{k^-}) \neq \emptyset$ $(1 \leq i \leq m)$ holds and hence $t_i = \mathbf{LUB}(\{q_{ij} \mid 0 \leq j \leq \gamma_i\})$ for some $t_i \in$

$\mathcal{T}(\mathcal{F} \cup \mathcal{Q})$ by Lemma 4.11. Thus, in Step 4 in Procedure 4.1, substitution ρ is defined as $\rho = \{x_i \mapsto t_i \mid 1 \le i \le m\} \cup \{x_i \mapsto t_i \mid x_i \in \mathcal{V}ar(l) \setminus \mathcal{V}ar(r), \langle t_i \rangle \in \mathcal{Q}^0_{final}\}$. Moreover, in **ADDTRANS** new states and transition rules are defined so that

$$l\rho' \vdash^*_{(k+1)^-} \langle l\rho \rangle \vdash_{k+1} q \tag{4.24}$$

where $\rho' = \{x_i \mapsto \langle t_i \rangle \mid 1 \le i \le n\}$. On the other hand, by Lemmas 4.6, 4.12 and (4.22), we have

$$x_i\sigma \vdash^*_{(k+1)^-} \langle t_i \rangle. \tag{4.25}$$

Summarizing (4.24) and (4.25), we obtain $l\sigma \vdash^*_{(k+1)^-} l\rho' \vdash^*_{k+1} q$ and the lemma holds. □

The next lemma shows the completeness of Procedure 4.1.

Lemma 4.16 *For two terms $s, s' \in \mathcal{T}(\mathcal{F})$ and a state $q \in \mathcal{Q}_0$, if $s' \vdash^*_0 q$ and $s \to^*_{I,\mathcal{R}} s'$, then there is an integer k such that $s \vdash^*_k q$.*

Proof. The proof is shown by induction on the number n of rewriting steps in $s \to^*_{I,\mathcal{R}} s'$. For the base case ($s = s'$), let $k = 0$, then the lemma holds. Assume the lemma holds for $n-1$ and consider the case with $n(\ge 1)$. The rewrite sequence of length n can be written as

$$s[o \leftarrow l\sigma] = s \to_{I,\mathcal{R}} s[o \leftarrow r\sigma] \to^*_{I,\mathcal{R}} s' \tag{4.26}$$

where $o \in \mathcal{P}os(t)$, σ is a substitution and $l \to r \in \mathcal{R}$. By the inductive hypothesis, there is an integer k such that $s[o \leftarrow r\sigma] \vdash^*_k q$ and hence there is a state q' such that

$$r\sigma \vdash^*_k q' \tag{4.27}$$

and

$$s[o \leftarrow q'] \vdash^*_k q. \tag{4.28}$$

From (4.26) and Lemma 2.1, we have

$$l\sigma \to_{I,\mathcal{R}} r\sigma. \tag{4.29}$$

By applying Lemma 4.15 to (4.27) and (4.29), it is possible that

$$l\sigma \vdash^*_{k+1} q'. \tag{4.30}$$

Summarizing (4.28) and (4.30), we have $s = s[o \leftarrow l\sigma] \vdash^*_{k+1} s[o \leftarrow q'] \vdash^*_k q$ and the lemma holds. □ Summarizing the lemmas in Section 4.3 and 4.4, the following theorem holds.

Theorem 4.17 *For an AO-TRS $\mathcal{R}$, if Procedure 4.1 halts with an output $\mathcal{A}_*$, then $\mathcal{L}(\mathcal{A}_*) = (\leftarrow^*_{I,\mathcal{R}})(NF_{\mathcal{R}})$.* □

Example 4.2 Consider the FPO^{-1}-TRS $\mathcal{R}_1$ in Example 3.1 again. $\mathcal{R}_1$ is an AO-TRS as well. Since $\mathcal{L}(\mathcal{A}_*) \supseteq \mathcal{L}(\mathcal{A}_2) = \mathcal{T}(\mathcal{F})$ by Example 4.1, we know $\mathcal{R}_1$ is SN by Theorem 4.1, Fact 4.2 and Theorem 4.17. □

4.5 Termination of Procedure 4.1

In this subsection, we will show if an AO-FPO^{-1}-TRS is given to Procedure 4.1, then the procedure always halts. For this purpose, we prove that there is an upper-bound on the number of states which are newly defined. Once the set of states saturates, then the set of transition rules also saturates and the procedure halts. First, as a measure of the size of a state, we define the *layer* of a state. Intuitively, the number of layers of a state is the number of left-hand sides of rewrite rules which are used to define the state. For a state $\langle t \rangle \in \mathcal{Q}_k$, define the number of *layers* of $\langle t \rangle$, denoted $\mathrm{layer}(\langle t \rangle)$, as follows; (1) if $t \in \mathcal{Q}$ or t is a ground subterm of l where $l \to r$ is a rewrite rule in $\mathcal{R}$, then $\mathrm{layer}(\langle t \rangle) = 0$ and (2) if $t = (l/o)\rho$ with $l \to r \in \mathcal{R}$, $o \in \mathcal{P}os(l)$, l/o not a variable, $\mathcal{V}ar(l/o) = \{x_1, \ldots, x_n\}$ and $\rho = \{x_i \mapsto t_i \mid 1 \leq i \leq n\}$, then $\mathrm{layer}(\langle t \rangle) = 1 + \max\{\mathrm{layer}(t_i) \mid 1 \leq i \leq n\}$. Remark that $\mathrm{layer}(\langle t \rangle)$ is not always uniquely determined by this definition. If different values are defined as $\mathrm{layer}(\langle t \rangle)$, then we choose the minimum among the values as $\mathrm{layer}(\langle t \rangle)$. The following lemma holds by the definition.

Lemma 4.18 *For states $\langle t \rangle, \langle t_1 \rangle, \ldots, \langle t_n \rangle \in \mathcal{Q}_k$, if $t = t_1 \sqcup \cdots \sqcup t_n$, then* $\mathrm{layer}(\langle t \rangle) = \max\{\mathrm{layer}(\langle t_i \rangle) \mid 1 \leq i \leq n\}$. □

We note that in the above definition, if $x_i \in \mathcal{V}ar(l) \setminus \mathcal{V}ar(r)$, then $t_i \in \mathcal{Q}$ (by Step 4 of Procedure 4.1) and $\mathrm{layer}(\langle t_i \rangle) = 0$. This means that variables which occurs only in the left-hand side of a rewrite rule are ignored when we consider the number of layers.

Example 4.3 For a state $q = \langle h(g(p_0)) \rangle$ in Example 4.1, $\mathrm{layer}(q) = 1$ since $q = h(g(x))\sigma$ with $\sigma = \{x \mapsto p_0\}$, $h(g(x))$ is the left-hand side of a rewrite rule in $\mathcal{R}_1$, and $\mathrm{layer}(p_0) = 0$. □

In the following, we show that if $\mathcal{R}$ is an AO-FPO^{-1}-TRS, then $\mathrm{layer}(q) \leq |\mathcal{R}|$ for any state $q \in \mathcal{Q}_k$ defined by Procedure 4.1 where $|\mathcal{R}|$ is the number of rewrite rules in $\mathcal{R}$. An outline of the proof is as follows. First we associate each rule in $\mathcal{R}$ with a non-negative integer called a *rank*. If $\mathcal{R}$ is an AO-FPO^{-1}-TRS, then the rank is well-defined and is less than $|\mathcal{R}|$. Next, it is shown that if a rewrite rule with rank j is used in Step 4 of Procedure 4.1, then $\mathrm{layer}(q) \leq j + 1$ for any state q which are defined in Step 4 in the same iteration of the loop. The *rank* of a rule in $\mathcal{R}$ is defined based on the sticking-out graph $G = (V, E)$ of $\mathcal{R}$. Let v be the vertex of G which corresponds to a rewrite rule $l \to r$ in $\mathcal{R}$. The *rank* of $l \to r$ is the maximum weight of a path to v from any vertex in V. If $\mathcal{R}$ is an FPO^{-1}-TRS, then the rank of any rewrite rule is a non-negative integer less than $|\mathcal{R}|$. For $\mathcal{R}_1$ in Example 3.1, the ranks of p_1 and p_2 are both zero, since there is no edge with weight one in the sticking-out graph of $\mathcal{R}_1$.

Lemma 4.19 *Let $l \to r$ be a rewrite rule and $\rho = \{x_i \mapsto t_i \mid 1 \leq i \leq m\} \cup \rho'$ be a substitution which are used in Step 4 of Procedure 4.1. If the rank of $l \to r$ is j, then layer$(\langle t_i \rangle) \leq j$ for each $1 \leq i \leq m$.* □

Before presenting a proof of the lemma, we first see how the number of layers of the state changes by a move of the TA. A transition rule of the TA $\mathcal{A}_k$ constructed in Procedure 4.1 is one of (a) a transition rule of $\mathcal{A}_0$, (b) a transition rule defined in **ADDTRANS**, or (c) a rewriting transition rule defined in Step 4.1 of Procedure 4.1. If a transition rule of (a) or (b) is used in a move, then the number of layers of the state associated with the head of the TA is increased by one or not changed by that move from the definition of the number of layers.

Proof of Lemma 4.19. The proof is by induction on the value of the loop variable k when a substitution ρ is used in Procedure 4.1. When $k = 0$, every state belongs to $\mathcal{Q}_0$ and layer$(\langle t_i \rangle) = 0$ for $1 \leq i \leq m$, and the lemma holds for any j. Assume that the lemma holds for $k \leq k' - 1$, and consider the case with $k = k'$. The inductive part is shown by contradiction. Without loss of generality, let $\langle t_1 \rangle$ be a state such that layer$(\langle t_1 \rangle) \geq j + 1$. The terms t_i must be given as $t_i = t_{i1} \sqcup \cdots \sqcup t_{i\gamma_i}$ by the function **LUB**. By Lemma 4.18, layer$(\langle t_1 \rangle) = \max\{\langle t_{1j} \rangle \mid 1 \leq j \leq \gamma_1\}$ holds. Let $\langle t_{11} \rangle$ be a state such that layer$(\langle t_1 \rangle) =$ layer$(\langle t_{11} \rangle)$ without loss of generality. Let us look at the sequence of moves

$$\hat{t} = r[o_{ij} \leftarrow q_{ij} \mid 1 \leq i \leq m, 1 \leq j \leq \gamma_i] \vdash_k^* q \tag{4.31}$$

considered in Step 4 and observe how the number of layers of the state changes as the head of $\mathcal{A}_k$ moves from the position o_{11} to the root position λ in $\hat{t}$: There are four different cases.

1. A rewriting transition rule is used at a certain position in $\hat{t}$. Let o be the innermost position among such positions. From the observation before the proof, there are two different subcases:
   (a) The number of layers does not increase at any o' with $o \prec o' \prec o_{11}$.
   (b) There is a position o' with $o \prec o' \prec o_{11}$ such that the number of layers increases at o'.
2. No rewriting transition rule is used from o_{11} to λ. There are two subcases:
   (a) The number of layers does not increase at any o' with $\lambda \prec o' \prec o_{11}$.

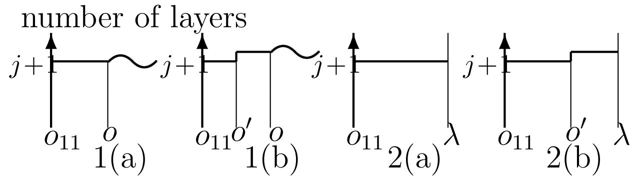

Figure 3: The number of layers of a state of $\mathcal{A}_k$ in (4.31).

(b) There is a position o' with $\lambda \prec o' \prec o_{11}$ such that the number of layers increases at o'.

These four cases are illustrated in Fig. 3.

Assume that the number of layers changes as in case 1(a) above. In this case we can derive a contradiction as follows (See [10] for a formal proof of a similar property). Consider the rewriting transition rule used by the first ϵ-move at the position o. Let $l' \to r'$ be the rewrite rule used for defining this rewriting transition rule in Step 4 of Procedure 4.1. Then, the state just before the rewriting move occurs at o can be written as $\langle l'\rho'' \rangle$ for some substitution ρ''. Remark that $\mathrm{layer}(\langle l'\rho'' \rangle) = \mathrm{layer}(\langle t_{11} \rangle) \geq j+1$ since the number of layers changes as in case 1(a). This implies that the substitution ρ'' replaces a variable in l' with a state which has j or more layers. Therefore, by using the inductive hypothesis, the rule $l' \to r'$ must have rank j or more. On the other hand, the fact that there are no rewriting moves at o' with $o \prec o' \prec o_{11}$ implies that l' properly sticks out of r/o as in condition 1 of the definition of sticking-out graph. Hence, the rank of $l \to r$ must be larger than the rank of $l' \to r'$, and consequently must be $j+1$ or more, a contradiction.

For case 2(a), it can be shown that there is a rewrite rule $l' \to r'$ with rank j or more, and a subterm of l' sticks out of r as in condition 2 of the definition of the sticking-out graph. For cases 1(b) and 2(b), it can be shown that there is a rewrite rule $l' \to r'$ with rank $j+1$ or more, and the rule satisfies conditions 3 and 4 of the definition of sticking-out graph, respectively. In either case, a contradiction is derived. Hence, the inductive part is shown and the proof completes. □

For an AO-FPO^{-1}-TRS $\mathcal{R}$, the rank of every rule is less than $|\mathcal{R}|$ and hence the number of layers of any state is $|\mathcal{R}|$ or less by Lemma 4.19 and the following lemma holds.

Lemma 4.20 *Procedure 4.1 halts if the input is an AO-FPO^{-1}-TRS.* □

In general, the running time of Procedure 4.1 is exponential to both of the size of the given TRS $\mathcal{R}$ and the size of the given TA $\mathcal{A}$.

4.6 Main Result

Lemma 4.21 *For an AO-FPO*$^{-1}$*-TRS* $\mathcal{R}$*, we can effectively construct a TA* $\mathcal{A}_*$ *such that* $\mathcal{L}(\mathcal{A}_*) = (\leftarrow^*_{I,\mathcal{R}})(NF_{\mathcal{R}})$.

Proof. By Theorem 4.17 and Lemma 4.20. □

By using this lemma, we can show the following main theorem.

Theorem 4.22 *For an AO-FPO*$^{-1}$*-TRS* $\mathcal{R}$ *the following problems are decidable. (1) Is* $\mathcal{R}$ *SN? (2) for a term* s*, is* s *SN in* $\mathcal{R}$*?*

Proof. By Lemmas 4.3 and 4.21. □

5 Conclusion

In the paper, we introduced AO-FPO^{-1}-TRS and we showed that SN property of the class is decidable (Theorem 4.22). In the proof, we adopted tree automata technique similar to the one in [13]. The class of AO-FPO^{-1}-TRSs properly includes AO-GR-TRSs[13, 15].

Right ground TRSs[5] and right-linear monadic TRSs[14] are also known to be TRSs whose SN property is decidable. Since these classes do not assume AO (almost orthogonal) property, the class of AO-FPO^{-1}-TRSs does not include them. On the other hand, a TRS $\mathcal{R}$ being an FPO^{-1}-TRS does not imply that $\mathcal{R}$ is right ground or right-linear monadic. Thus, the classes of FPO^{-1}-TRSs and right ground TRSs (right-linear monadic TRSs) do not have inclusion relation each other.

There are two directions for future work. The one is to find a wider class of TRSs whose SN is decidable. The proof of Theorem 4.22 is based on the idea that WIN and SN are equivalent for AO-TRSs. There are some classes which are wider than AO-TRSs and in which WIN and SN are equivalent[8].

Another direction is to extend the proof technique so that we can deal with TRS modulo equations. In [11], it is shown that SN for ground TRSs modulo AC is decidable.

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